Liouville Brownian motion

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Abstract

We construct a stochastic process, called the **Liouville Brownian motion** which we conjecture to be the scaling limit of random walks on large planar maps which are embedded in the Euclidean plane or in the sphere in a conformal manner. Our construction works for all universality classes of planar maps satisfying $\gamma < \gamma_c = 2$. In particular, this includes the interesting case of $\gamma = \sqrt{8/3}$ which corresponds to the conjectured scaling limit of large uniform planar p-angulations (with fixed $p \geqslant 3$).

We start by constructing our process from some fixed point $x \in \mathbb{R}^2$ (or $x \in \mathbb{S}^2$). This amounts to changing the speed of a standard two-dimensional brownian motion B_t depending on the local behaviour of the Liouville measure "$M_\gamma(dz) = e^{\gamma X} dz$" (where X is a Gaussian Free Field, say on $\mathbb{S}^2$). A significant part of the paper focuses on extending this construction simultaneously to all points $x \in \mathbb{R}^2$ or $\mathbb{S}^2$ in such a way that one obtains a semi-group $P_{\mathbf{t}}$ (the Liouville semi-group). We prove that the associated Markov process is a Feller diffusion for all $\gamma < \gamma_c = 2$ and that for all $\gamma < \gamma_c$, the Liouville measure M_γ is invariant under $P_{\mathbf{t}}$ (which in some sense shows that it is the right quantum gravity diffusion to consider).

This Liouville Brownian motion enables us to introduce a whole set of tools of stochastic analysis in Liouville quantum gravity, which will be hopefully useful in analyzing the Liouville quantum geometry. Also, we give sense to part of the celebrated Feynman path integrals which are at the root of Liouville quantum gravity, those related to integration along Brownian paths. Finally we believe that this work might shed some new light on the difficult problem of constructing a quantum metric out of the exponential of a Gaussian Free Field.

Key words or phrases: Liouville quantum gravity, Liouville Brownian motion, Gaussian multiplicative chaos.

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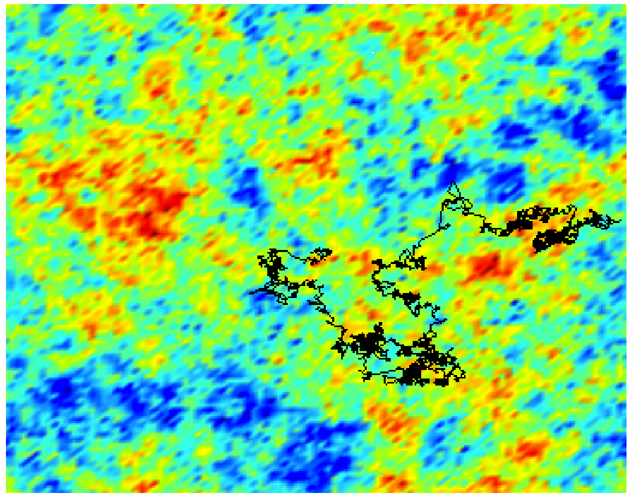

Figure 1: Simulation of a massive LBM on the unit torus. The background stands for the height landscape of the GFF on the torus: red for high values and blue for small values. The evolution of the LBM is plotted in black

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1 Introduction

Let $\mathcal{T}_n$ be the set of (conformally equivalent classes of) all triangulations of the two-dimensional sphere $\mathbb{S}^2$ with n faces with no loops or multiple edges. Choose uniformly at random a triangulation $T = T_n$ of $\mathcal{T}_n$ and consider a simple random walk on T. We want to identify the limit in law as $n \to \infty$ of this random walk. As stated, the problem seems to be ill-posed because there is some flexibility in the choice of the embedding of the triangulation into the sphere: we may apply a Möbius transformation of the sphere and obtain a conformally equivalent triangulation. As suggested in [12], one possible way to remove this flexibility is as follows. Consider a circle packing $P = (P_c)_{c\in C}$ in the sphere $\mathbb{S}^2$ such that the contact graph of P is T. This packing is unique up to Möbius transformations. In order to fix the Möbius transformation without spoiling the symmetries of the problem, one may consider among all circle packings P on the sphere $\mathbb{S}^2$ which realize the triangulation T, the packing P whose "barycenter" is the centre of the sphere. We refer to [12] for more

detail and for a justification through the Poincaré-Beardon and the Douady-Earle theorems that one can indeed find such a packing, which is unique up to rotations. This embedding will be denoted by P_T and constitutes a canonical discrete conformal structure for the triangulation T.

Now if one chooses uniformly at random a triangulation $T = T_n$ of $\mathcal{T}_n$ and if μ_T denotes the random probability measure on $\mathbb{S}^2$ which assigns a measure $1/n$ to each face of P_T, it is conjectured in [12] that μ_T converges in law towards a limiting probability measure given in terms of the Liouville measure

$$\text{“}M_{\sqrt{8/3}}(dz) = e^{\sqrt{8/3}X(z)}\,dz\text{”}\,,$$

where X is a Gaussian Free Field on the sphere $\mathbb{S}^2$ with vanishing mean (see subsection 3.1.1). Since there are several natural ways how to renormalize M_γ in order to obtain a probability measure on $\mathbb{S}^2$, a precise conjecture for the scaling limit of μ_T is still missing. Yet, in any case the limiting measure is believed to be absolutely continuous with respect to $M_{\sqrt{8/3}}$. See [12, 32, 38, 70] for more on this topic.

Conjecture 1. *Choose uniformly at random a triangulation T of $\mathcal{T}_n$ and consider a simple random walk on T. As n goes to ∞, the law of the simple random walk converges towards the law of the main object of this paper: the* **Liouville Brownian motion** *on the sphere for $\gamma^2 = \frac{8}{3}$.*

As such the purpose of this paper is to construct what can be thought of as the natural scaling limit of random walks on random triangulations (or similar models: quadrangulations, p-angulations, see [53, 54, 55, 56, 57, 58]) or, more generally, random walks on random triangulations weighted by the partition functions of suitable statistical physics models: Ising, $O(n)$, Potts models,...in which case, the γ in conjecture 1 must be adapted to the central charge of the model (see [34, 38, 40] for instance).

Nevertheless, the interest of the Liouville Brownian motion (LBM for short) that we are going to explain goes beyond the possibility of providing a natural candidate for scaling limits of random walks on planar maps. Indeed, recall that the ultimate mathematical problem in (critical) $2d$-Liouville quantum gravity is to construct a random metric on a two dimensional Riemannian manifold D, say a domain of $\mathbb{R}^2$ (or the sphere) equipped the Euclidean metric dz^2, which takes on the form

$$e^{\gamma X(z)}dz^2 \tag{1.1}$$

where X is a Gaussian Free Field (GFF) on the manifold D and $\gamma \in [0,2)$ is a coupling constant that can be expressed in terms of the central charge of the underlying model (see [51, 21] for further details and also [25, 26, 40, 60] for insights in Liouville quantum gravity). The simplicity of such an expression hides many highly non trivial mathematical difficulties. Indeed, the correlation function of a GFF presents a short scale logarithmically divergent behaviour that makes relation (1.1)

non rigorous. One has to apply a cutoff procedure to smooth down the singularity of the GFF and the method to do this at a "metric level" remains unclear. However, many geometric quantities are related to this metric and for some of them, the cutoff procedure may be applied properly without having a direct access to the metric. For instance, in a major conceptual step, the authors in [32] focused on the volume form M associated to the metric, sometimes called the Liouville measure. Their method falls under the scope of the theory of Gaussian multiplicative chaos developed by Kahane [44] (see also [66, 67] for a construction based on convolution techniques), which allows us to give a rigorous meaning to the expression

$$M(A)=\int_A e^{\gamma X(z)-\frac{\gamma^2}{2}\mathbb{E}[X(z)^2]}\,dz, \tag{1.2}$$

where dz stands for the volume form (Lebesgue measure) on D (to be exhaustive, one should integrate against $h(z)\,dz$ where h is a deterministic function involving the conformal radius at z but this term does not play an important role for our concerns). This strategy made possible an interpretation in terms of measures of the Knizhnik-Polyakov-Zamolodchikov formula (KPZ for short, see [51]) relating the fractal dimensions of sets as seen by the Lebesgue measure or the Liouville measure. The KPZ formula is proved in [32] when considering the fractal notion of expected box counting dimension whereas the fractal notion of almost sure Hausdorff dimension is considered in [65]. This measure based approach made also possible in [9] a mathematical understanding of *duality* in Liouville quantum gravity as well as a rigorous proof of the dual KPZ formula (see [3, 4, 19, 28, 29, 33, 34, 43, 48, 49, 50, 52] for an account of physics literature).

Another powerful tool in describing a Riemannian geometry is the Brownian motion. With it are attached several analytic objects serving to describe the geometry: semigroup, Laplace-Beltrami operator, heat kernel, Dirichlet forms... Therefore, a relevant way to have further insights into Liouville quantum gravity geometry is to define the *Liouville Brownian motion*. It can be constructed on any $2d$ background Riemannian manifold equipped with a Gaussian Free Field. However, in this paper and for pedagogical purposes, we will mostly describe the following situations: the whole plane $\mathbb{R}^2$, the sphere $\mathbb{S}^2$ or the torus $\mathbb{T}^2$, or planar bounded domains. Working on the whole plane $\mathbb{R}^2$ is convenient to avoid unnecessary complications related to stochastic calculus on manifolds. In that case, it is necessary to introduce a long scale infrared regulator in order to have a well defined field X. So we will consider a Massive Gaussian Free Field X on $\mathbb{R}^2$ (MFF for short). Then, we will describe the situation when the manifold is the sphere $\mathbb{S}^2$, the torus $\mathbb{T}^2$ or a planar bounded domain equipped with a GFF X (here compactness plays the role of infrared regulator). More generally, it is also clear that our methodology may apply to n-dimensional Riemannian manifold equipped with a log-correlated Gaussian field and yields similar results: we just work in dimension 2 because of motivations related to $2d$ Liouville quantum gravity.

Let us now comment on our results and explain the thread of our approach in the

situation of the whole plane $\mathbb{R}^2$ equipped with a MFF X. As previously explained, giving sense to (1.1) requires first to apply an ultraviolet cutoff to the MFF (this procedure is described in subsection 2.1) in order to smooth down the singularity of the covariance kernel. Let us denote by X_n the field resulting from a cutoff procedure at level n: we do not need to make this statement more precise now, let us just say that the field X has been smoothed up to some extent that is encoded by n: the larger n is, the closer to X the field X_n is. We can then consider a Riemannian

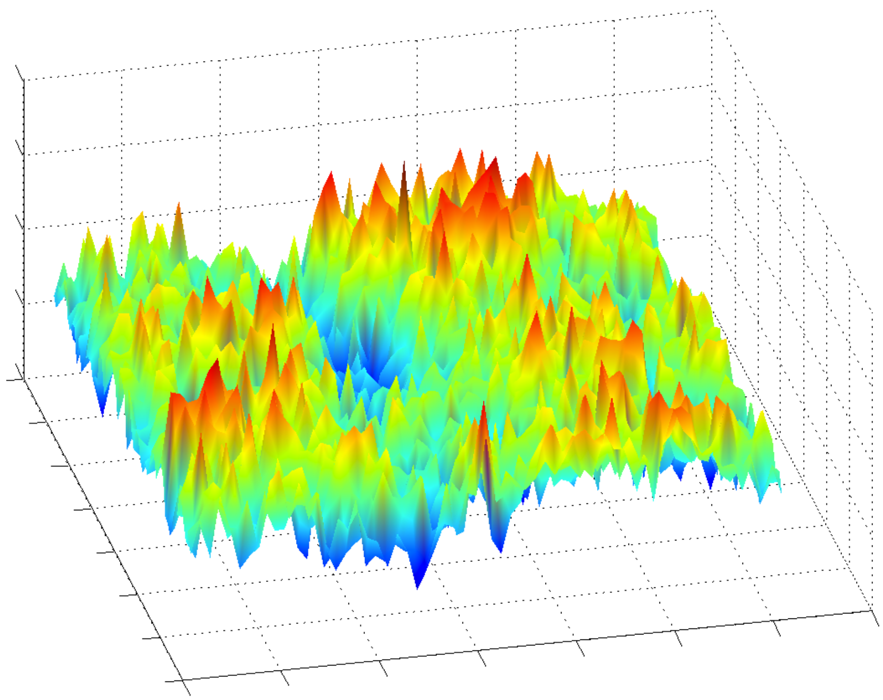

Figure 2: Simulation of a massive GFF on the unit torus.

metric tensor on $\mathbb{R}^2$:

$$g_n(z) = e^{\gamma X_n(z) - \frac{\gamma^2}{2}\mathbb{E}[X_n(z)^2]} dz^2, \tag{1.3}$$

which appears as a n-regularized form of (1.1). The renormalization term $e^{\frac{\gamma^2}{2}\mathbb{E}[X_n(z)^2]}$ appears for future renormalization purposes, which have the same origins as in (1.2), but does not play a role now in the geometry associated with this metric. The couple $(\mathbb{R}^2, g_n)$ is a Riemannian manifold. We further stress that the Riemannian volume of this manifold is nothing but the n-regularized version of (1.2):

$$M_n(A) = \int_A e^{\gamma X_n(z) - \frac{\gamma^2}{2}\mathbb{E}[X_n(z)^2]}\, dz,$$

which converges as $n \to \infty$ (meaning when the cutoff is removed) towards the Liouville measure M. One can associate to the Riemannian manifold $(\mathbb{R}^2, g_n)$ a Brownian motion $\mathcal{B}^n$ in a standard way: consider a standard 2-dimensional Brownian

motion $\bar{B}$ and define the n-regularized Liouville Brownian motion:

$$\begin{cases} \mathcal{B}^{n,x}_{\mathbf{t}=0} = x \\ d\mathcal{B}^{n,x}_{\mathbf{t}} = e^{-\frac{\gamma}{2}X_n(\mathcal{B}^{n,x}_{\mathbf{t}})+\frac{\gamma^2}{4}\mathbb{E}[X_n(\mathcal{B}^{n,x}_{\mathbf{t}})^2]}\,d\bar{B}_{\mathbf{t}}. \end{cases} \tag{1.4}$$

It is the solution of a SDE on $\mathbb{R}^2$. At first sight, it is not obvious to understand in which way the above n-regularized LBM will converge as $n \to \infty$. To get a better idea, let us use the Dambis-Schwarz theorem (or rather the Knight theorem in dimension 2) and rewrite (1.4) as

$$\mathcal{B}^{n,x}_{\mathbf{t}} \overset{law}{=} x + B_{\langle \mathcal{B}^{n,x}\rangle_{\mathbf{t}}}\,, \tag{1.5}$$

where $(B_r)_{r\geqslant 0}$ is another two-dimensional Brownian motion and the quadratic variation $\langle \mathcal{B}^{n,x}\rangle$ of $\mathcal{B}^{n,x}$ is given by:

$$\langle \mathcal{B}^{n,x}\rangle_{\mathbf{t}} := \inf\{s \geqslant 0\,:\, \int_0^s e^{\gamma X_n(x+B_u)-\frac{\gamma^2}{2}\mathbb{E}\left[X_n(x+B_u)^2\right]}\,du \geqslant \mathbf{t}\}\,. \tag{1.6}$$

Therefore, the n-regularized LBM appears as a time change of a standard Brownian motion and studying its convergence thus boils down to proving the convergence of its quadratic variations, which are entirely encoded by the mapping:

$$F_n(x,t) = \int_0^t e^{\gamma X_n(x+B_u)-\frac{\gamma^2}{2}\mathbb{E}\left[X_n(x+B_u)^2\right]}\,du. \tag{1.7}$$

More precisely, the quadratic variation of $\mathcal{B}^{n,x}$ is nothing but the inverse mapping of $F_n(x,\cdot)$. Notice also that $F_n(x,\cdot)$ can be seen as a Gaussian multiplicative chaos along the Brownian paths of B. Gaussian multiplicative chaos theory thus enters the game in order to establish the convergence of F_n. This can be done for all values of $\gamma \in [0,2)$. Nevertheless, as we will see, much more work is needed to deduce the convergence of the n-regularized LBM: we have to prove not only that $F_n(x,\cdot)$ converges towards a continuous, strictly increasing mapping $F(x,\cdot)$ but also to prove that this convergence holds, almost surely in X, for **all** the starting points $x \in \mathbb{R}^2$ in order to obtain a properly defined limiting Markov process. We will start with the following "one-point" case.

Theorem 1.1. *Assume $\gamma < 2$ and fix $x \in \mathbb{R}^2$. Almost surely in X and in B, the n-regularized Brownian motion $(\mathcal{B}^{n,x})_n$ defined by Definition 1.5 converges in the space $C(\mathbb{R}_+,\mathbb{R}^2)$ equipped with the supremum norm on compact sets towards a continuous random process $\mathcal{B}^x$, which we call (massive)* **Liouville Brownian motion** *starting from x, characterized by:*

$$\mathcal{B}^x_{\mathbf{t}} = x + B_{\langle \mathcal{B}^x\rangle_{\mathbf{t}}}$$

where $\langle \mathcal{B}^x\rangle$ is defined by

$$F(x,\langle \mathcal{B}^x\rangle_{\mathbf{t}}) = \mathbf{t}.$$

As a consequence, almost surely in X, the n-regularized Liouville Brownian motion defined in Definition 1.4 converges in law under $\mathbb{P}$ in $C(\mathbb{R}_+,\mathbb{R}^2)$ towards $\mathcal{B}^x$.

Before analyzing the convergence of the LBM "uniformly" in its starting point $x \in \mathbb{R}^2$, we will prove some a.s. properties satisfied by the above process when it starts from a fixed point $x \in \mathbb{R}^2$. For example we will see in subsection 2.5 that the process $(\mathcal{B}^x)_{\mathbf{t}\geqslant 0}$ a.s. does not get stuck on a point in $\mathbb{R}^2$ for a positive amount of time (typically on areas where the field X is very large). This boils down to proving that the limit of $F_n(x,\cdot)$ is a.s. continuous. This is slightly harder to see than the fact that $F=\lim_n F_n$ is a.s. strictly increasing since it requires a study of the moments of $F_n(x,\cdot)$. See subsection 2.6.

We will then extend the "one-point" case by showing that, almost surely in X, one can define the law of the Liouville Brownian motion simultaneously for all possible starting point $y \in \mathbb{R}^2$:

Theorem 1.2. *Assume $\gamma < 2$. Almost surely in X, for all $y \in \mathbb{R}^2$, the family $(F^n(y,\cdot))_n$ converges in law under $\mathbb{P}^B$ in $C(\mathbb{R}_+)$ equipped with the sup-norm topology towards a continuous increasing mapping $F(y,\cdot)$. Let us define the process $\mathbf{t}\mapsto\langle\mathcal{B}^y\rangle_{\mathbf{t}}$ by:*

$$\forall \mathbf{t}\geqslant 0,\quad F(y,\langle\mathcal{B}^y\rangle_{\mathbf{t}})=\mathbf{t}.$$

The law of the Liouville Brownian motion $\mathcal{B}^y$ starting from y is then given by

$$\mathcal{B}^y_{\mathbf{t}}=y+B_{\langle\mathcal{B}^y\rangle_{\mathbf{t}}}.$$

Almost surely in X, for all $y \in \mathbb{R}^2$, the process $\mathcal{B}^y$ is the limit in law in $C(\mathbb{R}_+)$ of the family $(\mathcal{B}^{n,y})_n$. Furthermore, almost surely in X and under $\mathbb{P}^B$, the law of the mapping $y\mapsto\mathcal{B}^y$ is continuous in $C(\mathbb{R}_+)$.

Therefore, the Liouville Brownian motion can be thought of as the solution of the following formal SDE

$$\begin{cases} \mathcal{B}^x_{\mathbf{t}=0}=x \\ d\mathcal{B}^x_{\mathbf{t}}=e^{-\frac{\gamma}{2}X(\mathcal{B}^x_{\mathbf{t}})+\frac{\gamma^2}{4}\mathbb{E}[X(\mathcal{B}^x_{\mathbf{t}})^2]}\,d\bar{B}_{\mathbf{t}}. \end{cases}\tag{1.8}$$

We will prove the following intriguing result. Once the environment X is fixed, the n-regularized Liouville Brownian motion $\mathcal{B}^{n,x}$ appearing in (1.4) is of course a measurable function of the Euclidean Brownian motion $\bar{B}$. We will prove not only that the couple $(\mathcal{B}^{n,x},\bar{B})$ converges in law towards the couple $(\mathcal{B}^x,\bar{B})$, but also that $\mathcal{B}^x$ is independent of $\bar{B}$. In a way, this can be interpreted as a creation of randomness by strongly pinching the Brownian curve $\bar{B}$ in order to create a new randomness $\mathcal{B}^x$ independent of $\bar{B}$.

It is natural to expect that the Liouville Brownian motion is Markovian. Actually, we can prove much more:

Theorem 1.3. *For $\gamma < 2$, the Liouville Brownian motion is a Feller Markov process with continuous sample paths.*

One may also view this Markov process as a semi-group. More precisely we prove along section 4 the following theorem.

Theorem 1.4. *For $\gamma < 2$, the Liouville Brownian motion induces a semi-group $(P^X_{\mathbf{t}})_{\mathbf{t}\geqslant 0}$ from $C_b(\mathbb{R}^2)$ into itself, called the Liouville semigroup. This semi-group is the limit as $n \to \infty$ of the regularized semi-groups $(P^n_{\mathbf{t}})_{\mathbf{t}\geqslant 0}$ (see Theorem 4.2).*

Finally, the Liouville semi-group P^X extends to a semi-group on $L^p(\mathbb{R}^2, M)$ for all $1 \leqslant p < \infty$ and it is strongly continuous for $1 < p < \infty$.

As a Markov process, it is natural to wonder whether the Liouville Brownian motion possesses an invariant measure:

Theorem 1.5. *For all $\gamma < 2$, the Liouville Brownian motion is reversible with respect to the Liouville measure. Therefore the Liouville measure is invariant. In particular, the Liouville semi-group P^X on $L^2(\mathbb{R}^2, M)$ is self-adjoint.*

This property hints that the Liouville Brownian motion is the right diffusion to consider if one wants to study the geometry of quantum gravity through the eyes of random walks and diffusions. This Markov process has a generator, which we call *Liouville Laplacian*, which formally reads

$$\Delta_X = e^{-\gamma X(x)+\frac{\gamma^2}{2}\mathbb{E}[X(x)^2]}\Delta.$$

and can be thought of as the Laplace-Beltrami operator of $2d$ Liouville quantum gravity.

We point out that, in the physics literature, we have tracked down the notions of heat kernel or Laplace-Beltrami operator of $2d$-Liouville quantum gravity at least in [4, 5, 22]. Diffusions related to quantum gravity are also investigated in [5, 17, 18, 20, 22, 76] (quoting all physics references is beyond the scope of this paper and certainly beyond our skills too) and this paper is mainly motivated by giving sense to the objects studied in [4, 22]. It may be worth pointing out that our methods allow us to give sense to Feynman path integrals of the type

$$\int_{C([0,T];\mathbb{R}^2)} f(\sigma)\exp\Big(-\frac{1}{2}\int_0^T e^{\gamma X(\sigma(s))-\frac{\gamma^2}{2}\mathbb{E}[X^2]}|\sigma'(s)|^2\,ds\Big)\mathcal{D}\sigma,$$

or

$$\int_{C([0,T];\mathbb{R}^2)} f(\sigma)\exp\Big(-\frac{1}{2}\int_0^T |\sigma'(s)|^2+\mu e^{\gamma X(\sigma(s))-\frac{\gamma^2}{2}\mathbb{E}[X^2]}\,ds\Big)\mathcal{D}\sigma$$

appearing throughout the physics literature.

Let us end this introduction by an informal discussion on the Dirichlet form associated to our *Liouville Brownian motion* as well as its possible relevance to the construction of the Liouville metric. First, the standard theory of symmetric Dirichlet forms (see [37] for instance) allows us to attach to the Liouville semigroup $(P^X_{\mathbf{t}})_{\mathbf{t}\geqslant 0}$ a Dirichlet form:

$$\Sigma(f,f) = \lim_{\mathbf{t}\to 0}\frac{1}{\mathbf{t}}\int \big(f(x)-P^X_{\mathbf{t}}f(x)\big)f(x)M(dx). \tag{1.9}$$

with domain $\mathcal{F}$, which is defined as the set of functions $f \in L^2(\mathbb{R}^2, M)$ for which the above limit exists and is finite. A thorough study of this Dirichlet form is carried out in [39], where the authors give a precise description of this Dirichlet form and prove that it is strongly local and regular. Let us point out that the technology from [37] is crucially used in [39]. This enables us to apply the full machinery of strongly local Dirichlet forms in Liouville quantum gravity. Coming back to the problem of the metric, over the last 20 years, a rich theory has been developed whose aim is to capture the "geometry" of the underlying space out of the Dirichlet form of a process living on that space. See for example [71, 72, 73]. This geometric aspect of Dirichlet forms can be interpreted in a sense as an extension of Riemannian geometry applicable to non differential structures. Among the recent progresses of Dirichlet forms has emerged the notion of *intrinsic metric* associated to a strongly regular Dirichlet form [15, 16, 23, 71, 72, 73, 75]. Interestingly, when the Riemannian tensor is smooth enough, the *intrinsic metric* matches the original Riemannian metric. It is thus natural to wonder if this theory is well suited to this problem of constructing the Liouville metric. If one can check that the topology associated to the intrinsic metric is Euclidean, the intrinsic metric is a proper distance and generates a non trivial length space. In [39], the authors prove that this intrinsic metric turns out to be 0, showing in a way that the Liouville geometry is far enough from the Riemannian standards, at least far enough to be out of reach from the intrinsic metric associated to a Dirichlet form. The reader may find further comments about the intrinsic metric in [39] as well as heuristics to understand why it vanishes.

We also stress that a stochastic analysis of the Liouville Brownian motion is carried out in the companion paper [39]. In particular, existence of the *Liouville heat kernel* is established:

Theorem 1.6 ([39])**.** *[Liouville heat kernel] For $\gamma < 2$, the Liouville semi-group $(P^X_{\mathbf{t}})_{\mathbf{t}}$ is absolutely continuous with respect to the Liouville measure and can therefore be written as*

$$P^X_{\mathbf{t}} f(x) = \int_{\mathbb{R}^2} f(y) p^X(x, y, \mathbf{t})\, M(dy)$$

for bounded continuous functions f. The family $(p^X(\cdot, \cdot, \mathbf{t}))_{\mathbf{t}}$ will be called Liouville heat kernel.

Finally, we stress that we are convinced that our approach opens many doors on this topic and, actually, raises many more questions than we can possibly address, at least in this paper. So, a whole section 5 is devoted to describing several related questions, with various ambition level.

Remark 1.7. *After posting online the first version of this manuscript, another work [14] on Liouville Brownian motion (starting from one point) appeared. The author's construction basically corresponds to the content of our Theorem 1.1.*

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Index of notations

- Liouville Brownian motion: $(\mathcal{B}_{\mathbf{t}})_{\mathbf{t}\geqslant 0}$
- Classical time: t v.s. Quantum time: $\mathbf{t}$

  We will thus distinguish the (quantum) time $\mathbf{t}$ along $\mathcal{B}_{\mathbf{t}}$ and the (classical) time along B_t.
- (massive) Gaussian Free Field: X.
- Liouville measure: $M = M_\gamma$.
- Space of continuous functions with compact support in a domain D: $C_c(D)$,
- Space of continuous functions vanishing at infinity on $\mathbb{R}^2$: $C_0(\mathbb{R}^2)$,
- Space of continuous bounded functions on D: $C_b(D)$,
- Space of continuous functions on $\mathbb{R}_+$: $C(\mathbb{R}_+)$ equipped with the sup-norm topology over compact sets.
- Laplace-Beltrami operator on a manifold: Δ

In what follows, we will consider Brownian motions B or $\bar{B}$ on $\mathbb{R}^2$ (or the sphere $\mathbb{S}^2$ or torus $\mathbb{T}^2$) independent of the underlying Free Field. We will denote by $\mathbb{E}^Y$ or $\mathbb{P}^Y$ expectations and probability with respect to a field Y. For instance, $\mathbb{E}^X$ or $\mathbb{P}^X$ (resp. $\mathbb{E}^B$ or $\mathbb{P}^B$) stand for expectation and probability with respect to the (M)GFF (resp. the Brownian motion B).

2 Liouville Brownian motion on the plane

In this section, we set out to construct the (massive) Liouville Brownian motion on the whole plane. As explained in introduction, we need to introduce an infrared regulator to get a well defined Free Field on the whole plane: put in other words, the massless GFF cannot be defined on the whole plane so that we have to apply a large scale cutoff. There are many ways of applying a large scale cutoff. Regarding physics literature, a natural way to do this is to consider a whole plane Massive Gaussian Free Field (MFF for short). So we first remind the reader of the construction of the MFF.

2.1 Massive Gaussian Free Field on the plane

We consider a whole plane Massive Gaussian Free Field (MFF) (see [41, 69] for an overview of the construction of the MFF and applications). It is a standard Gaussian in the Hilbert space defined as the closure of Schwartz functions over $\mathbb{R}^2$ with respect to the inner product

$$(f,g)_m = m^2(f,g)_2 - (f,\triangle g)_2,$$

where $(\cdot,\cdot)_2$ is the standard inner product on $L^2(\mathbb{R}^2)$. The real $m>0$ is called the mass. Its action on $L^2(\mathbb{R}^2)$ can be seen as a Gaussian distribution with covariance kernel given by the Green function G_m of the operator $m^2-\triangle$, i.e.:

$$(m^2-\triangle)G_m(x,\cdot) = 2\pi\delta_x.$$

The main differences with the GFF are that the MGFF can perfectly be defined on the whole plane since the massive term makes coercive the associated Dirichlet form $(\cdot,\cdot)_m$ on the plane: the mass term acts as a long-scale cutoff (or infrared cutoff/regulator). Furthermore, the MGFF does not possess conformal invariance properties.

It is a standard fact that the massive Green function can be written as an integral of the transition densities of the Brownian motion weighted by the exponential of the mass:

$$\forall x,y\in\mathbb{R}^2,\quad G_m(x,y) = \int_0^\infty e^{-\frac{m^2}{2}u-\frac{|x-y|^2}{2u}}\,\frac{du}{2u}. \tag{2.1}$$

Clearly, it is a kernel of σ-positive type in the sense of Kahane [44] since we integrate a continuous function of positive type with respect to a positive measure. It is furthermore a star-scale invariant kernel [2]: it can be rewritten as

$$G_m(x,y) = \int_1^{+\infty}\frac{k_m(u(x-y))}{u}\,du. \tag{2.2}$$

for some continuous covariance kernel k_m:

$$k_m(z) = \frac{1}{2}\int_0^\infty e^{-\frac{m^2}{2v}|z|^2-\frac{v}{2}}\,dv.$$

One can check that

$$G_m(x,y) = \ln_+\frac{1}{|x-y|} + g_m(x,y)$$

for some continuous and bounded function g_m, which decays exponentially fast to 0 when $|x-y|\to\infty$.

Let us consider an unbounded increasing sequence $(c_n)_{n\geqslant 1}$ such that $c_1=1$. For each $n\geqslant 1$, we consider a centered smooth Gaussian process Y_n with covariance kernel given by

$$\mathbb{E}[Y_n(x)Y_n(y)] = \int_{c_n}^{c_{n+1}}\frac{k_m(u(x-y))}{u}\,du.$$

The MGFF is the Gaussian distribution defined by

$$X(x)=\sum_{n\geqslant 1}Y_n(x)$$

where the processes $(Y_n)_n$ are assumed to be independent. We define the n-regularized field by

$$X_n(x)=\sum_{k=1}^{n}Y_k(x). \tag{2.3}$$

Actually, based on Kahane's theory of multiplicative chaos [44], the choice of the decomposition 2.3 will not play a part in the forthcoming results, excepted that it is important that the covariance kernel of X_n be smooth in order to associate to this field a Riemannian geometry.

2.2 n-regularized Riemannian geometry

We can consider a Riemannian metric tensor on $\mathbb{R}^2$ (using conventional notations in Riemannian geometry):

$$g_n=e^{\gamma X_n(x)-\frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}(dx_1^2+dx_2^2).$$

The factor $\gamma\geqslant 0$ is a parameter. The renormalization term $e^{\frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}$ appears for future renormalization purposes but does not play a role now in the geometry associated with this metric. Since the Riemannian manifold $(\mathbb{R}^2,g_n)$ is connected, it carries the structure of distance. More precisely, we denote by $DC(\mathbb{R}^2)$ the family of all parameterized differentiable curves $\sigma:[a,b]\to\mathbb{R}^2$. We will use the shorthand $\sigma:x\to y$ if $\sigma\in DC(\mathbb{R}^2)$ satisfies $\sigma(a)=x$ and $\sigma(b)=y$. Given $\sigma\in DC(\mathbb{R}^2)$, the length of σ is defined by

$$L_n(\sigma)=\int_a^b e^{\frac{\gamma}{2}X_n(\sigma_t)-\frac{\gamma^2}{4}\mathbb{E}[X_n(\sigma_t)^2]}|\sigma_t'|\,dt.$$

This definition is independent of the parameterization. The distance $d_n:\mathbb{R}^2\times\mathbb{R}^2\to [0,+\infty[$ is defined by

$$d_n(x,y)=\inf\big\{L_n(\sigma);\sigma\in DC(\mathbb{R}^2),\sigma:x\to y\big\}. \tag{2.4}$$

The topology induced by d_n on $\mathbb{R}^2$ coincides with the Euclidean topology. In particular, $(\mathbb{R}^2,d_n)$ is complete, which implies that the Riemannian manifold $(\mathbb{R}^2,g_n)$ is geodesically complete (by the Hopf-Rinow theorem).

The Riemannian volume on the manifold $(\mathbb{R}^2,g_n)$ is given by:

$$M_n(dx)=e^{\gamma X_n(x)-\frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}\,dx$$

and will be called n-regularized Liouville measure. Classical theory of Gaussian multiplicative chaos ensures that, almost surely in X, the family $(M_n)_n$ weakly converges towards a limiting Radon measure M, which is called the Liouville measure. The limiting measure is non trivial if and only if $\gamma \in [0,2)$. Concerning theory of Gaussian multiplicative chaos, the reader is referred to Kahane's original paper [44] (or [2] as the MGFF is star scale invariant). A few results of the theory are also gathered in Appendix A. We will denote by ξ_M the power law spectrum of M (see [2, 7, 65] for instance):

$$\forall p \geqslant 0, \quad \xi_M(p) = (2+\frac{\gamma^2}{2})p - \frac{\gamma^2}{2}p^2.$$

2.3 Definition of the n-regularized Brownian motion

One can associate to the Riemannian manifold $(\mathbb{R}^2, g_n)$ a Brownian motion $\mathcal{B}^n$. It is an intrinsic stochastic tool describing the geometry of the manifold:

Definition 2.1 (n-regularized Liouville Brownian motion)**.** *For any fixed $n \geqslant 1$, we define the following diffusion on $\mathbb{R}^2$. For any $x \in \mathbb{R}^2$,*

$$\begin{cases} \mathcal{B}^{n,x}_{\mathbf{t}=0} = x \\ d\mathcal{B}^{n,x}_{\mathbf{t}} = e^{-\frac{\gamma}{2}X_n(\mathcal{B}^{n,x}_{\mathbf{t}}) + \frac{\gamma^2}{4}\mathbb{E}[X_n(\mathcal{B}^{n,x}_{\mathbf{t}})^2]}\, d\bar{B}_{\mathbf{t}} \end{cases} \tag{2.5}$$

where $\bar{B}_t$ is a standard two-dimensional Brownian motion. Equivalently,

$$\mathcal{B}^{n,x}_{\mathbf{t}} = x + \int_0^{\mathbf{t}} e^{-\frac{\gamma}{2}X_n(\mathcal{B}^{n,x}_{\mathbf{u}}) + \frac{\gamma^2}{4}\mathbb{E}[X_n(\mathcal{B}^{n,x}_{\mathbf{u}})^2]}\, d\bar{B}_{\mathbf{u}}\,. \tag{2.6}$$

By using the Dambis-Schwarz Theorem, one can define the n-regularized Liouville Brownian motion as follows.

Definition 2.2. *For any $n \geqslant 1$ fixed and $x \in \mathbb{R}^2$,*

$$\mathcal{B}^{n,x}_{\mathbf{t}} = x + B_{\langle \mathcal{B}^{n,x}\rangle_{\mathbf{t}}}\,, \tag{2.7}$$

where $(B_r)_{r\geqslant 0}$ is a two-dimensional Brownian motion independent of the Massive Free Field X and where the **quadratic variation** *$\langle \mathcal{B}^{n,x}\rangle$ of $\mathcal{B}^{n,x}$ is defined as follows:*

$$\langle \mathcal{B}^{n,x}\rangle_{\mathbf{t}} := \inf\{s \geqslant 0\,:\, \int_0^s e^{\gamma X_n(x+B_u) - \frac{\gamma^2}{2}\mathbb{E}\left[X_n(x+B_u)^2\right]}\, du \geqslant \mathbf{t}\}\,. \tag{2.8}$$

Remark 2.3. *Note that in the two above equivalent definitions of the n-regularized LBM, the "driving" Brownian motions $\bar{B}_t$ and B_t are both independent of the Gaussian Free field X. Nevertheless, it is not correct that $(X, \bar{B}_t, B_t)$ are mutually independent. There is some dependency between (B_t) and $(\bar{B}_t)$ which depends on the field X and the value of $n \geqslant 1$. We shall see later that as the regularization $n \to \infty$, these two Brownian motions are asymptotically independent.*

Observe that (2.8) amounts to saying that the increasing process $\langle \mathcal{B}^{n,x} \rangle : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the differential equation:

$$\langle \mathcal{B}^{n,x} \rangle_{\mathbf{t}} = \int_0^{\mathbf{t}} e^{-\gamma X_n(x+B_{\langle \mathcal{B}^{n,x} \rangle_u}) + \frac{\gamma^2}{2}\mathbb{E}[X_n(x+B_{\langle \mathcal{B}^{n,x} \rangle_u})^2]}\, du. \tag{2.9}$$

We stress that $(\mathbb{R}^2, g_n)$ is a stochastically complete manifold. Mathematically, this means that

$$\forall \mathbf{t} \geqslant 0, \quad \mathbb{P}^B[\mathcal{B}^{n,x}_{\mathbf{t}} \in \mathbb{R}^2] = 1,$$

or equivalently that the Brownian motion $\mathcal{B}^n$ runs for all time (no killing effect).

Several standard facts can be deduced from the smoothness of X_n:

Proposition 2.4. *Let $n \geqslant 1$ be fixed. Clearly, since $x \in \mathbb{R}^2 \mapsto X_n(x)$ is a.s. (in X) a smooth function, the above n-regularized Liouville Brownian motion $\mathcal{B}^n$ a.s. induces a Feller diffusion on $\mathbb{R}^2$. Let us denote by $(P^n_{\mathbf{t}})_{\mathbf{t} \geqslant 0}$ its semi-group. Also, $\mathcal{B}^n$ is reversible with respect to the Riemannian volume M_n, which is therefore invariant for $\mathcal{B}^n$.*

Definition 2.5 (transition kernel)**.** *One has the existence of transition kernels $p^n(x, y, \mathbf{t})$ so that for any $f \in C_c(\mathbb{R}^2)$ and any $x \in \mathbb{R}^2$,*

$$P^n_{\mathbf{t}} f(x) = \int_{y \in \mathbb{R}^2} f(y) p^n(x, y, \mathbf{t}) dM_n(y)\,, \tag{2.10}$$

The transition kernel p^n is the minimal fundamental solution of

$$\partial_t p^n(t, x, \cdot) = \frac{1}{2} e^{-X_n(x) + \frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]} \triangle p^n(t, x\cdot), \quad \lim_{t \to 0} p^n(t, x, y) M_n(dy) = \delta_x(dy).$$

It is known that the heat kernel $p^n :]0, +\infty[\times \mathbb{R}^2 \times \mathbb{R}_2 \to]0, +\infty[$ is a positive C^∞-function. This follows from Hörmander's criterion for hypoellipticity [42]. Positivity is established in [1].

2.4 Convergence of the quadratic variations when starting from a given fixed point

Our first objective is to establish the convergence in law of the n-regularized Brownian motion. To this purpose, we first study the behavior of the quadratic variation $\langle \mathcal{B}^{n,x} \rangle_t$. It will be useful to define the following quantities:

Definition 2.6. *Let F^n be the following random function on $\mathbb{R}^2 \times \mathbb{R}_+$:*

$$F^n(x, s) = \int_0^s e^{\gamma X_n(x+B_u) - \frac{\gamma^2}{2}\mathbb{E}\left[X_n(x+B_u)^2\right]}\, du. \tag{2.11}$$

The interest of such quantity lies in the relation defining $\langle \mathcal{B}^{n,x} \rangle$, or equivalently by solving equation (2.9).

Lemma 2.7. *Fix $x \in \mathbb{R}^2$. The process $\langle \mathcal{B}^{n,x} \rangle$ is entirely characterized by:*

$$\int_0^{\langle \mathcal{B}^{n,x} \rangle_{\mathbf{t}}} e^{\gamma X_n(x+B_u) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x+B_u)^2]}\, du = \mathbf{t}.$$

Proof . By differentiating with respect to $\mathbf{t}$ the equation (2.9), we get:

$$\langle \mathcal{B}^{n,x} \rangle'_{\mathbf{t}} = e^{-\gamma X_n(B_{\langle \mathcal{B}^{n,x} \rangle_t}) + \frac{\gamma^2}{2}\mathbb{E}[X_n(B_{\langle \mathcal{B}^{n,x} \rangle_t})^2]}.$$

It is plain to deduce that:

$$\int_0^{\langle \mathcal{B}^{n,x} \rangle_{\mathbf{t}}} e^{\gamma X_n(x+B_u) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x+B_u)^2]}\, du = \mathbf{t}.$$

□

We now focus on the convergence of the quadratic variations when we consider only one arbitrary starting point.

Proposition 2.8. *Assume $\gamma^2 < 4$ and fix $x \in \mathbb{R}^2$. Almost surely in X and in B, the random measure*

$$F^n(x, dr) := e^{\gamma X_n(x+B_r) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x+B_r)^2]}\, dr$$

converges for the weak convergence of measures towards a random measure $F(x, dr)$ on $[0, +\infty[$ which has full support.

Proof. It is enough to prove the convergence on each interval $[0, T]$ for $T > 0$. The quantity $F^n(x, t)$ converges almost surely as $n \to \infty$ because it is a nonnegative martingale with respect to the parameter n. Let us prove that it is uniformly integrable.

Let us denote by ν_t the occupation measure of the Brownian motion B between 0 and t. Recall that for each bounded continuous function $f : \mathbb{R}^2 \to \mathbb{R}$

$$\int_{\mathbb{R}^2} f(x)\, d\nu_t(dx) = \int_0^t f(B_r)\, dr.$$

From Theorem A.4, we just have to prove that almost surely in B the measure ν_t is in the class R_α for $\alpha < 2$. This is a standard simple result, which we recall here for the sake of completeness. Further details on the topic may be found in [24] for instance. To be in the class R_α, it is enough to prove that for any $\alpha < 2$, the integral

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{|x-y|^\alpha}\, \nu_t(dx)\, \nu_t(dy)$$

is almost surely finite. It is enough to prove

$$\mathbb{E}^B\Big[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{|x-y|^\alpha}\, \nu_t(dx)\, \nu_t(dy)\Big] < +\infty.$$

So, let us compute this expectation:

$$\begin{aligned}\mathbb{E}^B\Big[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{|x-y|^\alpha}\,\nu_t(dx)\,\nu_t(dy)\Big] &= \mathbb{E}^B\Big[\int_0^t\int_0^t\frac{1}{|B_r-B_s|^\alpha}\,dr\,ds\Big]\\ &= \mathbb{E}^B\Big[\frac{1}{|B_1|^\alpha}\Big]\int_0^t\int_0^t\frac{1}{|r-s|^{\alpha/2}}\,dr\,ds\\ &< +\infty.\end{aligned}$$

This shows that, almost surely with respect to B, the measure ν_B is in the class R_α for $\alpha < 2$. Theorem A.4 implies that, for each measurable set A with finite ν_t-measure, the quantity

$$\nu_t^n(x,A) = \int_A e^{\gamma X_n(x+z)-\frac{\gamma^2}{2}\mathbb{E}[X_n(x+z)^2]}\,\nu_t(dz)$$

is a uniformly integrable martingale with respect to the parameter n. Therefore it converges almost surely towards a non trivial limit.

Since $\mathbb{R}^2$ has a finite ν_t-measure, we deduce that

$$F_n(x,t) = \nu_t^n(x,\mathbb{R}^2) \to \widetilde{\nu}_t(x,\mathbb{R}^2) \quad \text{as } n \to \infty.$$

In particular, we deduce that for a countable family of couples (s,t) with $s<t$ such that the intervals $[s,t]$ generate the Borel topology on $\mathbb{R}_+$, the family $(F^n(x,t) - F^n(x,s))_n$ converges almost surely towards a non trivial limit. Therefore, almost surely in X, the family $(F^n(x,dr))_n$ of random measures on $\mathbb{R}_+$ weakly converges towards a limiting random measure $F(x,dr)$.

Let us prove that the measure $F(x,dr)$ has full support. Let us consider an interval I. Obviously, almost surely in B, the event $\{F(x,I)>0\}$ is an event in the asymptotic sigma algebra generated by the random processes $(Y_n)_n$ (involved in the construction of X). Therefore a.s. in B, it has $\mathbb{P}^X$-probability 0 or 1. Since we have $\mathbb{P}^B$ a.s.

$$\mathbb{E}^X[F(x,I)] = \widetilde{\nu}_t(x,\mathbb{R}^2) = t,$$

it has $\mathbb{P}^X$-probability 1. Then we can consider a countable family $(I_n)_n$ of intervals generating the Borel sigma algebra on $\mathbb{R}_+$. Almost surely in X and B, we have $F(x,I_n)>0$ for all n. This shows that $F(x,dr)$ has full support, which equivalently means that the random mapping $t \mapsto F(x,t)$ is increasing, a.s. in X and B. $\square$

Remark 2.9. *Let us emphasize that Kahane's theory of Gaussian multiplicative chaos ensures that the law of the limiting mapping $F(x,\cdot)$ does not depend on the chosen regularization $(X_n)_n$ of X (see [44, 66]). Furthermore, in the case of a GFF X on a planar domain, this result is reinforced in [32] for the Liouville measure: the authors prove that circle average approximations of X and projections of X along the eigenvalues of the Laplacian yields almost surely the same Liouville measure. Though worth being written, one can adapt their argument to prove the corresponding result for the change of times F.*

2.5 The Liouville Brownian motion does not get stuck

In this section, we make sure that the Liouville Brownian motion does not get stuck in some area of the state space $\mathbb{R}^2$. Typically, this situation may happen over areas where the field X takes large values, therefore having as consequence to slow down the Liouville Brownian motion. Mathematically, this can be formulated as follows: check that the mapping $\mathbf{t} \mapsto F(x, \mathbf{t})$ is continuous and tends to ∞ as $\mathbf{t} \to \infty$.

Theorem 2.10. *Assume $\gamma^2 < 4$ and fix $x \in \mathbb{R}^2$. Almost surely in X and in B, the mapping*

$$F^n(x, \cdot) : \mathbf{t} \mapsto \int_0^{\mathbf{t}} e^{\gamma X_n(x+B_r) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x+B_r)^2]}\, dr$$

converges in the space $C(\mathbb{R}_+)$ towards the continuous increasing mapping $F(x, \cdot)$ on $[0, +\infty[$. Furthermore, a.s. in X and in B,

$$\lim_{\mathbf{t} \to \infty} F(x, \mathbf{t}) = +\infty. \tag{2.12}$$

Proof. Let us first prove that the measure $F(x, dr)$ has no atom, which equivalently means that the random mapping $t \mapsto F(x, t)$ is continuous, a.s. in X and B. It suffices to prove that it has no atom on each interval $[0, T]$ for each $T > 0$. Without loss of generality, we may assume that $T = 1$. Also, it suffices to prove that $F(x, dr)$ has no atom of size $\delta > 0$ for all $\delta > 0$. Observe that

$$\{F(x, \cdot) \text{ has an atom of size } \delta\} \subset \bigcap_{n \geqslant 1} \bigcup_{k=0}^{n-1} \{F(x, [\frac{k}{n}, \frac{k+1}{n}]) \geqslant \delta\}.$$

Using the Markov inequality, it is enough to show:

$$\lim_{n \to \infty} \sum_{k=0}^{n-1} \mathbb{E}^B \mathbb{E}^X \Big(F(x, [\frac{k}{n}, \frac{k+1}{n}])^q \Big) = 0$$

for some $q > 0$. Let us admit for a while that Proposition 2.12 of subsection 2.6 below is true. Then, it suffices to choose any $q \in]1, 4/\gamma^2[$ because $\xi(q) > 1$ for such a q, i.e.:

$$\sum_{k=0}^{n-1} \mathbb{E}^B \mathbb{E}^X \Big(F(x, [\frac{k}{n}, \frac{k+1}{n}])^q \Big) \leqslant C_q n^{1-\xi(q)} \to 0, \quad \text{as } n \to \infty.$$

Now that we have proved that the mapping $F(x, \cdot)$ is continuous and increasing, the Dini theorem ensures that $(F^n(x, \cdot))_n$ converges in $C(\mathbb{R}_+)$ equipped with the sup-norm topology over compact sets towards $F(x, \cdot)$.

Finally, we prove that

$$\lim_{t \to \infty} F(x, t) = +\infty.$$

We give two proofs of this fact. A first simple proof in the case $0 \leqslant \gamma^2 < 2$ and a more elaborate proof to treat the general case $0 \leqslant \gamma^2 < 4$. Furthermore, we stress that the $0 \leqslant \gamma^2 < 2$ proof yields a stronger result on the asymptotic behaviour of F so that, in a way, the two proofs do not overlap.

Proof 1. (simple case) We will use the weak law of large numbers (WLLN) for covariance stationary sequences of random variables. Let us set

$$W_n = F(x,]n, n+1]).$$

We have $\mathbb{E}^X \mathbb{E}^B[W_n] = 1$, and

$$\begin{aligned}
\mathbb{E}^X \mathbb{E}^B[W_n W_{n+k}] &= \mathbb{E}^B \int_n^{n+1} \int_{n+k}^{n+k+1} e^{\gamma^2 G_m(B_r, B_u)} \, dr \, du \\
&= \int_0^1 \int_k^{k+1} \mathbb{E}^B \big[e^{\gamma^2 G_m(B_r, B_u)} \big] \, dr \, du.
\end{aligned}$$

To apply the WLLN theorem, we must check that

$$\frac{1}{n} \sum_{k=0}^{n} \mathbb{E}^X \mathbb{E}^B[W_0 W_k] \to 0, \quad \text{as } n \to \infty. \tag{2.13}$$

We have (for some constant C independent of n, which may vary along lines):

$$\begin{aligned}
\sum_{k=0}^{n} \mathbb{E}^X \mathbb{E}^B[W_0 W_k] &\leqslant C \int_0^1 \int_0^{n+1} \mathbb{E}^B \Big[\frac{1}{|B_r - B_u|^{\gamma^2}} \Big] \, dr du \\
&= C \int_0^1 \int_0^{n+1} \mathbb{E}^B \Big[\frac{1}{|B_1|^{\gamma^2}} \Big] \frac{1}{|r-u|^{\frac{\gamma^2}{2}}} \, dr du \\
&= C \sum_{k=1}^{n} \frac{1}{k^{\gamma^2/2}}.
\end{aligned}$$

It is plain to deduce that criterion (2.13) holds. We deduce that in $\mathbb{P}^X \otimes \mathbb{P}^B$-probability

$$\lim_{n \to \infty} \frac{1}{n} F(x, n) = 1.$$

Since $F(x, \cdot)$ is increasing, it is plain to deduce that its limit as $t \to \infty$ is ∞.

Proof 2. We consider the following sequence of stopping times associated to the Brownian motion:

$$T_n = \inf\{t \geqslant 0, \, |B_t| = n\}, \quad \bar{T}_n = \inf\{t \geqslant T_n, \, |B_t - B_{T_n}| = \frac{1}{4}, \}$$

We consider a subsequence $(n_j)_{j \geqslant 1}$ such that the following property holds for all $l \leqslant k$:

$$\sum_{l \leqslant j < j' \leqslant k} \alpha_{j,j'} \leqslant k - l$$

where $\alpha_{j,j'} = \sup_{|x| \leqslant n_j, |y| \geqslant n_{j'}} G_m(x,y)$.

We get:

$$
\begin{aligned}
&\mathbb{P}^B\mathbb{P}^X(\cap_{l \leqslant j \leqslant k}(F(x,]T_{n_j},\bar{T}_{n_j}]) \leqslant c)) \\
&\leqslant c^{k-l}\mathbb{E}^X\mathbb{E}^B\big[\Pi_{l \leqslant j \leqslant k}\frac{1}{F(x,]T_{n_j},\bar{T}_{n_j}])}\big] \\
&= c^{k-l}\mathbb{E}^X\mathbb{E}^B\Big[\frac{1}{\int_{]T_{n_l},\bar{T}_{n_l}]\times\cdots\times]T_{n_k},\bar{T}_{n_k}]} e^{\gamma(X(B_{s_l})+\cdots+X(B_{s_k}))-\frac{\gamma^2}{2}(\mathbb{E}[X(B_{s_l})^2]+\cdots+\mathbb{E}[X(B_{s_k})^2])} ds_l \dots ds_k}\Big] \\
&\leqslant c^{k-l}\mathbb{E}^X\mathbb{E}^B\Big[\frac{1}{\int_{]T_{n_l},\bar{T}_{n_l}]\times\cdots\times]T_{n_k},\bar{T}_{n_k}]} e^{\gamma(X(B_{s_l})+\cdots+X(B_{s_k}))-\frac{\gamma^2}{2}\mathbb{E}[(X(B_{s_l})+\cdots+X(B_{s_k}))^2]} ds_l \dots ds_k}\Big].
\end{aligned}
$$

Now, if we introduce $\bar{X}$ the free field with a cutoff (say $\mathbb{E}[\bar{X}_x\bar{X}_y] = \ln_+ \frac{1}{|y-x|}$), then by Kahane's inequality (Lemma A.5), we get that (let Y be a standard Gaussian variable):

$$
\begin{aligned}
&\mathbb{P}^B\mathbb{P}^X\big(\cap_{l \leqslant j \leqslant k}\{F(x,]T_{n_j},\bar{T}_{n_j}]) \leqslant c\}\big) \\
&\leqslant c^{k-l}\mathbb{E}^X\mathbb{E}^B\Big[\frac{1}{\int_{]T_{n_l},\bar{T}_{n_l}]\times\cdots\times]T_{n_k},\bar{T}_{n_k}]} e^{\gamma(X(B_{s_l})+\cdots+X(B_{s_k}))-\frac{\gamma^2}{2}\mathbb{E}[(X(B_{s_l})+\cdots+X(B_{s_k}))^2]}}\Big] \\
&\leqslant c^{k-l}\mathbb{E}^{\bar{X}}\mathbb{E}^B\Big[\frac{1}{\int_{]T_{n_l},\bar{T}_{n_l}]\times\cdots\times]T_{n_k},\bar{T}_{n_k}]} e^{\gamma(\bar{X}(B_{s_l})+\cdots+\bar{X}(B_{s_k}))-\frac{\gamma^2}{2}\mathbb{E}[(\bar{X}(B_{s_l})+\cdots+\bar{X}(B_{s_k}))^2]}}\Big] \\
&\quad\times \mathbb{E}[\frac{1}{e^{\gamma\sqrt{k-l}Y-\frac{\gamma^2}{2}(k-l)}}] \\
&\leqslant (ce^{\gamma^2})^{k-l}\mathbb{E}^{\bar{X}}\mathbb{E}^B\Big[\frac{1}{\int_{]T_{n_1},\bar{T}_{n_1}]} e^{\gamma\bar{X}(B_{s_1})-\frac{\gamma^2}{2}\mathbb{E}[(\bar{X}(B_{s_1})^2]}}\Big]^{k-l}.
\end{aligned}
$$

Observe that the latter expectation is finite (see Lemma 2.13 below). One then chooses c such that

$$
ce^{\gamma^2}\mathbb{E}^{\bar{X}}\mathbb{E}^B[\frac{1}{\int_{]T_{n_1},\bar{T}_{n_1}]} e^{\gamma\bar{X}(B_{s_1})-\frac{\gamma^2}{2}\mathbb{E}^{\bar{X}}[(\bar{X}(B_{s_1})^2]}}] < 1.
$$

By letting k go to infinity, we conclude that:

$$
\mathbb{P}^B\mathbb{P}^X(\cap_{l \leqslant j \leqslant \infty}(F(x,]T_{n_j},\bar{T}_{n_j}]) \leqslant c)) = 0.
$$

Hence, we get that:

$$
\mathbb{P}^B\mathbb{P}^X(\cap_l \cup_{l \leqslant j \leqslant \infty}(F(x,]T_{n_j},\bar{T}_{n_j}]) > c)) = 1.
$$

Since

$$
\lim_{t\to\infty} F(x,t) \geqslant c\sum_{j \geqslant 1} 1_{\{F(x,]T_{n_j},\bar{T}_{n_j}])>c\}},
$$

the proof is complete. □

Corollary 2.11. *Assume $\gamma^2 < 4$ and fix $x \in \mathbb{R}^2$. Almost surely in X and in B, the family $(B, \langle \mathcal{B}^{n,x} \rangle)_n$ converges in the space $C(\mathbb{R}_+, \mathbb{R}^2) \times C(\mathbb{R}_+, \mathbb{R}_+)$ equipped with the supremum norm on compact sets towards the couple $(B, \langle \mathcal{B}^x \rangle)$, characterized by:*

$$\forall \mathbf{t} \geqslant 0, \quad F(x, \langle \mathcal{B}^x \rangle_{\mathbf{t}}) = \mathbf{t}.$$

As such, the mapping $\mathbf{t} \mapsto \langle \mathcal{B}^x \rangle_{\mathbf{t}}$ is defined on $\mathbb{R}_+$, continuous and increasing.

Proof of Corollary 2.11. Almost surely in X and B, the family $(F^n(x, \cdot))_n$ converges in $C(\mathbb{R}_+, \mathbb{R}_+)$ towards $F(x, \cdot)$. Since

$$F^n(x, \langle \mathcal{B}^{n,x} \rangle_t) = t,$$

and

$$\lim_{t \to \infty} F(x, t) = +\infty,$$

it is plain to deduce that the family $(\langle \mathcal{B}^{n,x} \rangle)_n$ converges in $C(\mathbb{R}_+, \mathbb{R}_+)$ towards a continuous increasing process characterized by:

$$F(x, \langle \mathcal{B}^x \rangle_{\mathbf{t}}) = \mathbf{t}. \quad \square$$

2.6 Study of the moments

Proposition 2.12. *If $\gamma^2 < 4$ and $x \in \mathbb{R}^2$, the mapping $F(x, \cdot)$ possesses moments of order $0 \leqslant q < \min(2, 4/\gamma^2)$.*

Furthermore, if F admits moments of order $q \geqslant 1$ then, for all $s \in [0,1]$ and $t \in [0, T]$:

$$\mathbb{E}^X \mathbb{E}^B [F(x, [t, t+s])^q] \leqslant C_q s^{\xi(q)},$$

where $C_q > 0$ (independent of x, T) and

$$\xi(q) = \big(1 + \frac{\gamma^2}{4}\big) q - \frac{\gamma^2}{4} q^2.$$

Proof. For pedagogical purposes, we give here a short argument to prove the finiteness of the moments when $\gamma^2 < 2$. There is here no exception to the rule in multiplicative chaos that studying finiteness of the moments beyond the L^2 threshold is much more involved. So, the whole Appendix B is devoted to investigating the general case (in particular $2 \leqslant \gamma^2 < 4$).

Given a point $x \in \mathbb{R}^2$, we have:

$$\begin{aligned}
&\limsup_{n \to \infty} \mathbb{E}^X \mathbb{E}^B [F^n(x,t)^2] \\
&\qquad = \limsup_{n \to \infty} \mathbb{E}^X \mathbb{E}^B \Big[\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\gamma X_n(x+u) + \gamma X_n(x+v) - \frac{\gamma^2}{2} \mathbb{E}[X_n(x+u)^2] - \frac{\gamma^2}{2} \mathbb{E}[X_n(x+v)^2]} \Big] \nu_t(du) \nu_t(dv) \\
&\qquad = \mathbb{E}^B \Big[\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\gamma^2 G_m(u-v)} \nu_t(du) \nu_t(dv) \Big] \\
&\qquad = \mathbb{E}^B \Big[\int_0^t \int_0^t e^{\gamma^2 G_m(B_r - B_s)} ds dr \Big] \\
&\qquad < +\infty
\end{aligned}$$

when $\gamma^2 < 2$.

Now we assume that $\gamma^2 < 4$ and that F possesses moments of order $q \geqslant 1$. We prove the estimate concerning the power law spectrum. We first prove it when $t = 1$ and starting from $x \in \mathbb{R}^2$ and then we deduce the uniform estimate in t when starting from x.

$$\begin{aligned}
F(x,s) &= \int_0^s e^{\gamma X(x+B_r)-\frac{\gamma^2}{2}\mathbb{E}[X(x+B_r)^2]}\,dr \\
&= s\int_0^1 e^{\gamma X(x+B_{us})-\frac{\gamma^2}{2}\mathbb{E}[X(x+B_{us})^2]}\,du \\
&\overset{law}{=} s\int_0^1 e^{\gamma X(\sqrt{s}B_u)-\frac{\gamma^2}{2}\mathbb{E}[X(\sqrt{s}B_u)^2]}\,du. \qquad (2.14)
\end{aligned}$$

From star scale invariance of X (see Appendix A for a brief reminder about star scale invariance and further references), we have

$$G_m(\sqrt{s}u,\sqrt{s}v) \leqslant \ln\frac{1}{\sqrt{s}} + G_m(u,v).$$

Then, by taking the q-th power and expectation in (2.14) and Kahane's convexity inequalities (see Lemma A.5) together with the above domination, we get

$$\begin{aligned}
\mathbb{E}^X\mathbb{E}^B[F(x,s)^q] &\leqslant s^q\mathbb{E}^B\mathbb{E}^X\Big[\Big(e^{\gamma\Omega_{\sqrt{s}}-\frac{\gamma^2}{2}\mathbb{E}[\Omega_{\sqrt{s}}^2]}\int_0^1 e^{\gamma X(B_u)-\frac{\gamma^2}{2}\mathbb{E}[X(B_u)^2]}\,du\Big)^q\Big] \\
&= s^q\mathbb{E}^B\mathbb{E}^X\Big[e^{q\gamma\Omega_{\sqrt{s}}-q\frac{\gamma^2}{2}\mathbb{E}[\Omega_{\sqrt{s}}^2]}\Big]\mathbb{E}^B\mathbb{E}^X\Big[\Big(\int_0^1 e^{\gamma X(B_u)-\frac{\gamma^2}{2}\mathbb{E}[X(B_u)^2]}\,du\Big)^q\Big] \\
&= C_q s^{\xi(q)}
\end{aligned}$$

where $\Omega_{\sqrt{s}}$ is a Gaussian random variable with mean 0 and variance $\ln\frac{1}{\sqrt{s}}$ and independent of $\big(\int_0^t e^{\gamma X(B_u)-\frac{\gamma^2}{2}\mathbb{E}[X(B_u)^2]}\,du\big)_t$, and $C_q = \mathbb{E}^B\mathbb{E}^X\Big[\Big(\int_0^1 e^{\gamma X(B_u)-\frac{\gamma^2}{2}\mathbb{E}[X(B_u)^2]}\,du\Big)^q\Big]$ is independent of s, x.

Now we treat the general case.

$$\begin{aligned}
F(x,[t,t+s]) &= \int_t^{t+s} e^{\gamma X(x+B_r)-\frac{\gamma^2}{2}\mathbb{E}[X(x+B_r)^2]}\,dr \\
&= \int_0^s e^{\gamma X(x+B_t+\bar{B}_r)-\frac{\gamma^2}{2}\mathbb{E}[X(x+B_t+\bar{B}_r)^2]}\,dr
\end{aligned}$$

where $\bar{B}_r = B_{r+t} - B_t$ for $r \geqslant 0$ is a Brownian motion starting from 0 and indepen-

dent of $(x+B_u)_{u\leqslant t}$. We deduce by stationarity of the field X:

$$\begin{aligned}
\mathbb{E}^X\mathbb{E}^B[F(x,[t,t+s])^q]&\\
&=\mathbb{E}^B\mathbb{E}^X\Big[\Big(\int_0^s e^{\gamma X(x+B_t+\bar{B}_r)-\frac{\gamma^2}{2}\mathbb{E}[X(x+B_t+\bar{B}_r)^2]}\,dr\Big)^q\Big]\\
&=\mathbb{E}^B\mathbb{E}^X\Big[\Big(\int_0^s e^{\gamma X(\bar{B}_r)-\frac{\gamma^2}{2}\mathbb{E}[X(\bar{B}_r)^2]}\,dr\Big)^q\Big]\\
&\leqslant C_q s^{\xi(q)}.
\end{aligned}$$

The proof is complete. Let us stress that, for $q\in[0,1]$, the same argument as above yields

$$\mathbb{E}^X\mathbb{E}^B[F(x,[t,t+s])^q]\geqslant C_q s^{\xi(q)}.$$

□

Now we investigate finiteness of moments of negative order:

Lemma 2.13. *Let us denote by T_r the first exit time of the Brownian motion B out of the disk of radius $r\in]0,1]$ centered at x. For all $q>0$, there exists some constant $C>0$ (depending on q) such that:*

$$\sup_{n\geqslant 0}\mathbb{E}^X\mathbb{E}^B\Big[\Big(\frac{1}{\int_0^{T_r}e^{\gamma X_n(x+B_s)-\frac{\gamma^2}{2}\mathbb{E}\big[X_n(x+B_s)^2\big]}\,ds}\Big)^q\Big]\leqslant C\Big(\frac{1}{r}\Big)^{2q+\frac{q(1+q)}{2}\gamma^2}. \qquad (2.15)$$

Proof. Without loss of generality, we can take $x=0$ by stationarity of the field X. Furthermore, from Kahane's convexity inequalities, it suffices to prove the result for one log-correlated Gaussian field. Let us choose the exact scale invariant field $\bar{X}$ with covariance kernel given by:

$$\mathbb{E}[\bar{X}(x)\bar{X}(y)]=\ln_+\frac{1}{|x-y|}.$$

Let us also consider a white noise decomposition $(\bar{X}_\epsilon)_{\epsilon\in]0,1]}$ of $\bar{X}$ as constructed in [66]. In particular, the process $\epsilon\to\bar{X}_\epsilon$ has independent increments and $\bar{X}_{\epsilon,\epsilon'}:=\bar{X}_\epsilon-\bar{X}_{\epsilon'}$ has a correlation cutoff of length ϵ' (i.e. if the Euclidean distance between two sets A,B is greater than ϵ' then $(\bar{X}_{\epsilon,\epsilon'}(x))_{x\in A}$ and $(\bar{X}_{\epsilon,\epsilon'}(x))_{x\in B}$ are independent). The correlation structure of $(\bar{X}_\epsilon)_{\epsilon\in]0,1]}$ is given for $\epsilon\in]0,1]$ by:

$$\mathbb{E}[\bar{X}_\epsilon(x)\bar{X}_{\epsilon'}(y)]=\begin{cases}0 & \text{if } |x-y|>1\\ \ln\frac{1}{|x-y|} & \text{if } \epsilon\leqslant|x-y|\leqslant 1\\ \ln\frac{1}{\epsilon}+2\big(1-\frac{|x-y|^{1/2}}{\epsilon^{1/2}}\big) & \text{if } |y-x|\leqslant\epsilon\end{cases}.$$

Therefore, we have to prove

$$\sup_{\epsilon\in]0,1]}\mathbb{E}^{\bar{X}}\mathbb{E}^B\Big[\Big(\frac{1}{\int_0^{T_r}e^{\gamma\bar{X}_\epsilon(x+B_s)-\frac{\gamma^2}{2}\mathbb{E}\big[\bar{X}_\epsilon(x+B_s)^2\big]}\,ds}\Big)^q\Big]\leqslant C\Big(\frac{1}{r}\Big)^{2q+\frac{q(1+q)}{2}\gamma^2}. \qquad (2.16)$$

Recall that the supremum is reached for $\epsilon \to 0$ by the martingale property. Now, if $\widetilde{T}_{\frac{1}{4}}$ is the first time $B_{t+T_{3/4}} - B_{T_{3/4}}$ hits the disk of radius $\frac{1}{4}$, we get:

$$\begin{aligned}
&\int_0^{T_1} e^{\gamma \bar{X}_\varepsilon(B_s) - \frac{\gamma^2}{2}\ln\frac{1}{\varepsilon}}\, ds \\
&\geqslant \int_0^{T_{\frac{1}{4}}} e^{\gamma \bar{X}_\varepsilon(B_s) - \frac{\gamma^2}{2}\ln\frac{1}{\varepsilon}}\, ds + \int_0^{\widetilde{T}_{\frac{1}{4}}} e^{\gamma \bar{X}_\varepsilon(B_{s+T_{3/4}}) - \frac{\gamma^2}{2}\ln\frac{1}{\varepsilon}}\, ds \\
&\geqslant e^{\gamma \inf_{|x|\leqslant 1} \bar{X}_{1/4}(x) - \frac{\gamma^2}{2}\ln 4} \left(\int_0^{T_{\frac{1}{4}}} e^{\gamma \bar{X}_{\varepsilon,1/4}(B_s) - \frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\, ds + \int_0^{\widetilde{T}_{\frac{1}{4}}} e^{\gamma \bar{X}_{\varepsilon,1/4}(B_{s+T_{3/4}}) - \frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\, ds \right).
\end{aligned}$$

The main observation is that, under the annealed measure $\mathbb{E}^{\bar{X}}\mathbb{E}^B$, the above two integrals are independent random variables. Indeed, we get for two functionals F, G:

$$\begin{aligned}
&\mathbb{E}^{\bar{X}}\mathbb{E}^B\Big[F\left(\int_0^{T_{\frac{1}{4}}} e^{\gamma \bar{X}_{\varepsilon,1/4}(B_s) - \frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\, ds\right) G\left(\int_0^{\widetilde{T}_{\frac{1}{4}}} e^{\gamma \bar{X}_{\varepsilon,1/4}(B_{s+T_{3/4}}) - \frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\, ds\right)\Big] \\
&= \mathbb{E}^B\Big[\mathbb{E}^{\bar{X}}[F\left(\int_0^{T_{\frac{1}{4}}} e^{\gamma \bar{X}_{\varepsilon,1/4}(B_s) - \frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\, ds\right)]\mathbb{E}^{\bar{X}}[G\left(\int_0^{\widetilde{T}_{\frac{1}{4}}} e^{\gamma \bar{X}_{\varepsilon,1/4}(B_{s+T_{3/4}}) - \frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\, ds\right)]\Big] \\
&= \mathbb{E}^B\Big[\mathbb{E}^{\bar{X}}[F\left(\int_0^{T_{\frac{1}{4}}} e^{\gamma \bar{X}_{\varepsilon,1/4}(B_s) - \frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\, ds\right)]\mathbb{E}^{\bar{X}}[G\left(\int_0^{\widetilde{T}_{\frac{1}{4}}} e^{\gamma \bar{X}_{\varepsilon,1/4}(B_{s+T_{3/4}} - B_{T_{3/4}}) - \frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\, ds\right)]\Big] \\
&= \mathbb{E}^{\bar{X}}\mathbb{E}^B\Big[F\left(\int_0^{T_{\frac{1}{4}}} e^{\gamma \bar{X}_{\varepsilon,1/4}(B_s) - \frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\, ds\right)\Big]\mathbb{E}^{\bar{X}}\mathbb{E}^B\Big[G\left(\int_0^{\widetilde{T}_{\frac{1}{4}}} e^{\gamma \bar{X}_{\varepsilon,1/4}(B_{s+T_{3/4}} - B_{T_{3/4}}) - \frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\, ds\right)\Big] \\
&= \mathbb{E}^{\bar{X}}\mathbb{E}^B\Big[F\left(\int_0^{T_{\frac{1}{4}}} e^{\gamma \bar{X}_{\varepsilon,1/4}(B_s) - \frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\, ds\right)\Big]\mathbb{E}^{\bar{X}}\mathbb{E}^B\Big[G\left(\int_0^{\widetilde{T}_{\frac{1}{4}}} e^{\gamma \bar{X}_{\varepsilon,1/4}(B_{s+T_{3/4}}) - \frac{\gamma^2}{2}\ln\frac{1}{4\varepsilon}}\, ds\right)\Big],
\end{aligned}$$

where we have used the fact that $\bar{X}_{\varepsilon,1/4}$ has a correlation cutoff of length $1/4$ for the first equality and the fact that the field $\bar{X}_{\varepsilon,1/4}$ is stationary for the second equality.

For all $r \in]0,1]$, we have:

$$\begin{aligned}
\int_0^{T_r} e^{\gamma \bar{X}_{r\varepsilon}(B_s) - \frac{\gamma^2}{2}\ln\frac{1}{r\varepsilon}}\, ds &= r^2 \int_0^{\frac{T_r}{r^2}} e^{\gamma \bar{X}_{r\varepsilon}(r\frac{B_{r^2 s'}}{r}) - \frac{\gamma^2}{2}\ln\frac{1}{r\varepsilon}}\, ds' \\
&= r^2 \int_0^{\widetilde{T}_1} e^{\gamma \bar{X}_{r\varepsilon}(r\widetilde{B}_{s'}) - \frac{\gamma^2}{2}\ln\frac{1}{r\varepsilon}}\, ds'
\end{aligned}$$

where $\widetilde{B}_{s'} = \frac{B_{r^2 s'}}{r}$ is a Brownian motion and $\widetilde{T}_1 = \frac{T_r}{r^2}$ is the first time it hits the disk of radius 1. Therefore, we get the following scaling relation in distribution for all $r \in]0,1]$ under the annealed measure:

$$\int_0^{T_r} e^{\gamma \bar{X}_{r\varepsilon}(B_s) - \frac{\gamma^2}{2}\ln\frac{1}{r\varepsilon}}\, ds \overset{(law)}{=} r^2 e^{\gamma \Omega_r - \frac{\gamma^2}{2}\ln\frac{1}{r}} \int_0^{T_1} e^{\gamma \bar{X}_\varepsilon(B_s) - \frac{\gamma^2}{2}\ln\frac{1}{\varepsilon}}\, ds \tag{2.17}$$

where Ω_r is independent from B, $(\bar{X}_\epsilon)_\epsilon$ and with distribution $N(0, \ln\frac{1}{r})$. From this scaling relation and the above considerations, we deduce that we can find some variable N with negative moments and such that we have the following stochastic domination:

$$Y \geqslant N(Y_1 + Y_2)$$

where (Y_1, Y_2) are i.i.d. of distribution Y and independent from N where Y is distributed like $\lim_{\varepsilon\to 0} \int_0^{T_1} e^{\gamma \bar{X}_\varepsilon(B_s) - \frac{\gamma^2}{2}\ln\frac{1}{\varepsilon}}\, ds$. Then we get from adapting [59] that:

$$\sup_{\varepsilon > 0} \mathbb{E}^{\bar{X}} \mathbb{E}^B \Big[\Big(\frac{1}{\int_0^{T_1} e^{\gamma \bar{X}_\varepsilon(B_s) - \frac{\gamma^2}{2}\ln\frac{1}{\varepsilon}}\, ds} \Big)^q \Big] < \infty$$

One then deduces inequality (2.16) from (2.17). □

From this, one easily deduces the following lemma:

Lemma 2.14. *For all $q > 0$ and all $t \in]0,1]$, there exists some constant $C > 0$ (depending on q) such that:*

$$\sup_{n \geqslant 0} \mathbb{E}^X \mathbb{E}^B \Big[\Big(\frac{1}{\int_0^t e^{\gamma X_n(x+B_s) - \frac{\gamma^2}{2}\mathbb{E}\big[X_n(x+B_s)^2\big]}\, ds} \Big)^q \Big] \leqslant C \Big(\frac{1}{t}\Big)^{q + \frac{q(1+q)}{4}\gamma^2}. \tag{2.18}$$

Proof. Without loss of generality, we can take $x = 0$ by stationarity of the field X. Furthermore, by using the same techniques as the proof of proposition 2.12, we just have to show the inequality for $t = 1$. We have (recall that T_r denote the first exit times):

$$\frac{1}{\big(\int_0^1 e^{\gamma X(B_s) - \frac{\gamma^2}{2}\mathbb{E}\big[X(B_s)^2\big]}\, ds\big)^q} \leqslant \sum_{n=1}^{\infty} \frac{1_{\{T_{1/2^n} \leqslant 1 \leqslant T_{1/2^{n-1}}\}}}{\big(\int_0^{T_{1/2^n}} e^{\gamma X(B_s) - \frac{\gamma^2}{2}\mathbb{E}\big[X(B_s)^2\big]}\, ds\big)^q}.$$

Therefore, we get

$$\begin{aligned}
&\mathbb{E}^{X,B}\Big[\Big(\int_0^1 e^{\gamma X(B_s) - \frac{\gamma^2}{2}\mathbb{E}\big[X(B_s)^2\big]}\, ds\Big)^{-q}\Big] \\
&\leqslant \sum_{n=1}^{\infty} \mathbb{E}^{X,B}\Big[1_{\{T_{1/2^n} \leqslant 1 \leqslant T_{1/2^n}\}} \Big(\int_0^{T_{1/2^n}} e^{\gamma X(B_s) - \frac{\gamma^2}{2}\mathbb{E}\big[X(B_s)^2\big]}\, ds\Big)^{-q}\Big] \\
&\leqslant \sum_{n=1}^{\infty} \mathbb{P}\big(T_{1/2^{n-1}} \geqslant 1\big)^{1/2} \mathbb{E}^{X,B}\Big[\Big(\int_0^{T_{1/2^n}} e^{\gamma X(B_s) - \frac{\gamma^2}{2}\mathbb{E}\big[X(B_s)^2\big]}\, ds\Big)^{-2q}\Big]^{1/2} \\
&\leqslant C \sum_{n=1}^{\infty} e^{-c 2^{2n}} (2^n)^{2q + \frac{q(1+2q)}{2}\gamma^2} \\
&< \infty.
\end{aligned}$$

The proof is complete. □

2.7 Convergence of the quadratic variations for all points

Now we want to define almost surely in X the change of times $F(y,\cdot)$ for all starting points $y \in \mathbb{R}^2$. The task is not obvious because most of the desired properties of $F(y,\cdot)$ can be established "almost surely in X" for a given fixed point. Therefore, apart from the obvious situation when one considers a countable quantity of points $y \in \mathbb{R}^2$, the properties of $F(y,\cdot)$ cannot be assumed to hold simultaneously for an uncountable quantity of points $y \in \mathbb{R}^2$. We briefly draw below the picture of our strategy:

1. First we prove that almost surely in X, under $\mathbb{P}^B$ the sequence $(F^n(y,\cdot))$ is tight in $C(\mathbb{R}_+)$ simultaneously for all possible starting points $y \in \mathbb{R}^2$,
2. From Theorem 2.10, we further know that it converges for a countably dense sequence of points of $\mathbb{R}^2$.
3. We prove some uniform continuity estimates with respect to y and we deduce that the limit $F(y,\cdot)$ is continuous with respect to y (in some sense that we will make precise later).
4. Finally, we deduce that its inverse $\langle \mathcal{B}^y \rangle$ is also continuous w.r.t. y.

Now we come down into details. In what follows, we will assume that, almost surely in X, under $\mathbb{P}^B$ the sequence $(F^n(y,\cdot))$ converges in $C(\mathbb{R}_+)$ for all possible rational starting points $y \in \mathbb{Q}^2$.

In what follows, if B is a Brownian motion on $\mathbb{R}^2$, we will denote by B^1 and B^2 its components. We will further use throughout this subsection, the following coupling lemma, the proof of which is standard and left to the reader.

Lemma 2.15. *Fix $y_0 \in \mathbb{R}^2$ and let us start a Brownian motion B^{y_0} from y_0. Let us consider another independent Brownian motion B starting from 0 and denote by B^y, for a rational $y \in \mathbb{R}^2$, the Brownian motion $B^y = y + B$. Let us denote by τ_1^y the first time at which the first components of B^{y_0} and B^y coincide:*

$$\tau_1^y = \inf\{u > 0; B_u^{1,y_0} = B_u^{1,y}\}$$

and by τ_2^y the first time at which the second components coincide after τ_1^y:

$$\tau_2^y = \inf\{u > \tau_1; B_u^{2,y_0} = B_u^{2,y}\}$$

The random process $\overline{B}^{y,y_0}$ defined by

$$\overline{B}_t^{y,y_0} = \begin{cases} (B_t^{1,y_0}, B_t^{2,y_0}) & \text{if } t \leqslant \tau_1 \\ (B_t^{1,y}, B_t^{2,y_0}) & \text{if } \tau_1 < t \leqslant \tau_2 \\ (B_t^{1,y}, B_t^{2,y}) & \text{if } \tau_2 < t. \end{cases}$$

is a new Brownian motion on $\mathbb{R}^2$ starting from y_0, and coincides with B^y for all times $t > \tau_2^y$. Furthermore, as $y \to y_0$, we have for all $\eta > 0$:

$$\mathbb{P}(\tau_2^y > \eta) \to 0.$$

Now we claim:

Proposition 2.16. *Almost surely in X, for all $y_0 \in \mathbb{R}^2$, under $\mathbb{P}^B$, the family $(F^n(y_0,\cdot))_n$ is tight in $C(\mathbb{R}_+)$.*

Proof. We first say a few words about the main ingredient of the proof: controlling logarithmic singularities with respect to the Liouville measures M_n. This stems from the first observation that:

$$\mathbb{E}^B\big(F^n(y_0,T)\big) = \int_{\mathbb{R}^2}\int_0^T e^{-\frac{|u-y_0|^2}{2s}}\frac{ds}{2\pi s}M_n(du),$$

and, second, that

$$\int_0^T e^{-\frac{|u-y_0|^2}{2s}}\frac{ds}{2\pi s} \simeq C\ln\frac{1}{|u-y_0|}, \quad \text{for } |u-y_0|\to 0.$$

Now we handle the proof rigorously. It will be based on the three following lemmas, which together will easily lead to the proof of Proposition 2.16.

Lemma 2.17. *Almost surely in X, for all $R>0$ and $T>0$:*

$$\lim_{A\to\infty}\sup_{n\geqslant 1}\sup_{y\in B(0,R)}\mathbb{P}^B\Big(F^n(y,T)\geqslant A\Big)=0. \tag{2.19}$$

Lemma 2.18. *Almost surely in X, for all $y_0\in\mathbb{R}^2$, for each fixed $0<T'<T$:*

$$\forall\eta>0,\quad \limsup_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}^B\Big(\sup_{\substack{T'\leqslant s,t\leqslant T\\ |t-s|\leqslant\delta}}|F^n(y_0,t)-F^n(y_0,s)|\geqslant\eta\Big)=0. \tag{2.20}$$

Lemma 2.19. *Almost surely in X, for all $R>0$,*

$$\lim_{T\to 0}\sup_{n\geqslant 1}\sup_{y\in B(0,R)}\int_{\mathbb{R}^2}\int_0^T e^{-\frac{|u-y|^2}{2s}}\frac{ds}{2\pi s}M_n(du)=0. \tag{2.21}$$

We postpone the proof of the above lemmas and finish first the proof of Proposition 2.16. By combining the three above lemmas, we deduce that, almost surely in X, for all $y_0\in\mathbb{R}^2$, the sequence $(F^n(y_0,\cdot))_n$ is tight in $C([0,+\infty[)$ equipped with the sup norm topology over compact sets. Indeed, we have for all $\eta>0$ and $0<T'<T$:

$$\begin{aligned}
&\mathbb{P}^B\Big(\sup_{\substack{0\leqslant s,t\leqslant T\\ |t-s|\leqslant\delta}}|F^n(y_0,t)-F^n(y_0,s)|\geqslant\eta\Big)\\
&\leqslant \mathbb{P}^B\big(2\sup_{0\leqslant s\leqslant T'}F^n(y_0,s)\geqslant\frac{\eta}{2}\big)+\mathbb{P}^B\Big(\sup_{\substack{T'\leqslant s,t\leqslant T\\ |t-s|\leqslant\delta}}|F^n(y_0,t)-F^n(y_0,s)|\geqslant\frac{\eta}{2}\Big)\\
&=\mathbb{P}^B\big(F^n(y_0,T')\geqslant\frac{\eta}{4}\big)+\mathbb{P}^B\Big(\sup_{\substack{T'\leqslant s,t\leqslant T\\ |t-s|\leqslant\delta}}|F^n(y_0,t)-F^n(y_0,s)|\geqslant\frac{\eta}{2}\Big)\\
&\leqslant\frac{4}{\eta}\mathbb{E}^B\big(F^n(y_0,T')\big)+\mathbb{P}^B\Big(\sup_{\substack{T'\leqslant s,t\leqslant T\\ |t-s|\leqslant\delta}}|F^n(y_0,t)-F^n(y_0,s)|\geqslant\frac{\eta}{2}\Big).
\end{aligned} \tag{2.22}$$

Therefore, we can take the $\limsup_{\delta\to 0}\limsup_{n\to\infty}$ in (2.22) and apply Lemma 2.18 to get

$$\limsup_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}^B\Big(\sup_{\substack{0\leqslant s,t\leqslant T\\|t-s|\leqslant\delta}}|F^n(y_0,t)-F^n(y_0,s)|\geqslant\eta\Big)$$
$$\leqslant\frac{4}{\eta}\limsup_n\mathbb{E}^B\big(F^n(y_0,T')\big). \tag{2.23}$$

Now, observe that

$$\mathbb{E}^B\big(F^n(y_0,T')\big)=\int_{\mathbb{R}^2}\int_0^{T'}e^{-\frac{|u-y_0|^2}{2s}}\frac{ds}{2\pi s}M_n(du).$$

From Lemma 2.19, we deduce

$$\lim_{T'\to 0}\sup_{n\geqslant 1}\sup_{y\in B(0,R)}\mathbb{E}^B\big(F^n(y,T')\big)=0,$$

in such a way that the right-hand side of (2.23) can be made arbitrarily close to 0 when T' goes to 0. $\square$

Now we carry out the proof of the three lemmas.

Proof of Lemma 2.18. We use the coupling procedure of Lemma 2.15, consider the law of $F^n(y_0,\cdot)$ constructed with the Brownian motion $\overline{B}^{y,y_0}$ and find $y\in\mathbb{Q}^2$ such that $\mathbb{P}^B(\tau_2^y\geqslant T')\leqslant\epsilon$. We have:

$$\begin{aligned}
&\mathbb{P}^{\overline{B}^{y,y_0}}\Big(\sup_{\substack{T'\leqslant s,t\leqslant T\\|t-s|\leqslant\delta}}|F^n(y_0,t)-F^n(y_0,s)|\geqslant\eta\Big)\\
&\leqslant\mathbb{P}^{\overline{B}^{y,y_0}}\Big(\sup_{\substack{T'\leqslant s,t\leqslant T\\|t-s|\leqslant\delta}}|F^n(y_0,t)-F^n(y_0,s)|\geqslant\eta;\tau_2^y\geqslant T'\Big)\\
&\quad+\mathbb{P}^{\overline{B}^{y,y_0}}\Big(\sup_{\substack{T'\leqslant s,t\leqslant T\\|t-s|\leqslant\delta}}|F^n(y_0,t)-F^n(y_0,s)|\geqslant\eta;\tau_2^y<T'\Big)\\
&\leqslant\epsilon+\mathbb{P}^{B^y}\Big(\sup_{\substack{T'\leqslant s,t\leqslant T\\|t-s|\leqslant\delta}}|F^n(y,t)-F^n(y,s)|\geqslant\eta\Big).
\end{aligned}$$

Since $y\in\mathbb{Q}^2$, the family $(F^n(y,\cdot))_n$ is tight in $C(\mathbb{R}_+,\mathbb{R}_+)$ with respect to the uniform topology over compact sets. Therefore we have

$$\limsup_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}^{B^y}\Big(\sup_{\substack{T'\leqslant s,t\leqslant T\\|t-s|\leqslant\delta}}|F^n(y,t)-F^n(y,s)|\geqslant\eta\Big)=0.$$

The proof is complete. $\square$

Proof of Lemma 2.17. It suffices to prove that almost surely in X,

$$\sup_{n \geqslant 1} \sup_{y \in B(0,R)} \mathbb{E}^B[F^n(y,T)] < +\infty.$$

Once again, we observe that

$$\mathbb{E}^B[F^n(y,T)] = \int_{\mathbb{R}^2} \int_0^T e^{-\frac{|u-y|^2}{2s}} \frac{ds}{2\pi s} M_n(du),$$

and therefore it suffices to prove that:

$$\sup_{n \geqslant 1} \sup_{y \in B(0,R)} \int_{\mathbb{R}^2} \int_0^T e^{-\frac{|u-y|^2}{2s}} \frac{ds}{2\pi s} M_n(du) < +\infty. \tag{2.24}$$

Let us first analyze the case where $|u-y| \leqslant \sqrt{T}$.
We will use the following fact.

Lemma 2.20. *For any $R > 0$ and for any $t \geqslant 0$, let $p_R(t,0,x)$ be the heat-kernel of the Brownian motion starting at 0 and killed the first time it exists the ball $B(0,R)$. We also denote by $p(t,0,y) = \frac{1}{2\pi t} e^{-|x|^2/2t}$ the usual heat-kernel on $\mathbb{R}^2$. If $R > 1$, then there exists a constant $C_R < \infty$ such that for any $t \in [0,1]$ and any x in the unit disk $\mathbb{D}$,*

$$p(t,0,x) \leqslant C_R \, p_R(t,0,x)\,.$$

Proof: This is a classical fact for which we did not find a reference. One way to see why this holds is to express $p(t,0,x)$ as the limit as $\varepsilon \to 0$ of the probability that the standard Brownian motion starting at 0 arrives at distance less than ε from x, renormalised by $\pi\varepsilon^2$. Letting $\varepsilon \to 0$, we obtain that $p_R(t,0,x)$ is exactly $p(t,0,x)$ times the probability $A_{t,R}$ that the Brownian bridge $(W_s)_{0 \leqslant s \leqslant t}$ from 0 to x remains in $B(0,R)$. By rewriting the Brownian bridge $(W_s)_{0 \leqslant s \leqslant t}$ as $\sqrt{t}$ times a Brownian bridge $(\widetilde{W_s})_{0 \leqslant s \leqslant 1}$ from 0 back to 0 plus a deterministic linear interpolation, one can easily see that $B_R := \inf_{t\in[0,1]} A_{t,R} > 0$, which ends the proof by taking $C_R = B_R^{-1}$. □

By applying this fact with $R = 2$ (and $C = C_2 < \infty$) together with a scaling argument, one thus obtains for any $0 \leqslant s \leqslant T$ and any point $u \in B(y, \sqrt{T})$:

$$p(s,y,u) \leqslant C \, p_{B(y,2\sqrt{T})}(s,y,u)$$

Integrating this inequality for $s \in [0,T]$, we obtain that for any y, u s.t. $|u-y| \leqslant \sqrt{T}$,

$$\int_0^T e^{-\frac{|u-y|^2}{2s}} \frac{ds}{2\pi s} \leqslant C \, G_{B(0,2\sqrt{T})}(0, u-y) = C \log \frac{2\sqrt{T}}{|u-y|}\,.$$

We now need to control the following quantity (possibly also as a function of $T > 0$ in order to handle Lemma 2.19 as well):

$$\sup_{n \geqslant 1} \sup_{y \in B(0,R)} \int_{|u-y| \leqslant \sqrt{T}} C \, \log \frac{2\sqrt{T}}{|u-y|} M_n(du)\,. \tag{2.25}$$

Let us first suppose $T < 1$. We divide the ball $B(y, \sqrt{T})$ into annulus of radii $2^{-k}\sqrt{T}, k \geqslant 0$. This gives us

$$\begin{aligned}\int_{|u-y| \leqslant \sqrt{T}} C \log \frac{2\sqrt{T}}{|u-y|} M_n(du) &\leqslant \sum_{k \geqslant 0} C \log 2^{k+2} M_n(B(y, 2^{-k}\sqrt{T})) \\ &\leqslant \widetilde{C} \sum_{k \geqslant 0} (k+2)(\sqrt{T} 2^{-k})^{\alpha/2} \\ &= \widetilde{C}\, T^{\alpha/4} \sum_{k \geqslant 0} (k+2)\, 2^{-\alpha k/2}\end{aligned}$$

where, by Theorem 2.22 below, $\widetilde{C}$ is some random constant which does not depend on y or n and $\alpha > 0$ is a strictly positive constant.

If on the other hand $T > 1$, then we use the fact that if $y \in B(0, R)$:

$$\begin{aligned}&\int_{|u-y| \leqslant \sqrt{T}} \log \frac{2\sqrt{T}}{|u-y|} M_n(du) \\ &\qquad \leqslant \int_{|u-y| \leqslant 1} \log \frac{2\sqrt{T}}{|u-y|} M_n(du) + \log(2\sqrt{T})\, M_n(B(0, R+\sqrt{T}))\,.\end{aligned}$$

The first term is analyzed exactly as above, and the second term is a u.i. martingale. As such the part of the integral in (2.24) with $|u-y| \leqslant \sqrt{T}$ is handled. (And one also notices that we will easily obtain Lemma 2.19 this way by letting $T \to 0$).

It remains to treat the case where $|u-y| > \sqrt{T}$.

By a standard change of variables, we have:

$$\begin{aligned}\int_0^T e^{-\frac{|u-y|^2}{2s}} \frac{ds}{2\pi s} = \int_0^{\frac{T}{|u-y|^2}} e^{-\frac{1}{2s}} \frac{ds}{2\pi s} &= \int_{\frac{|u-y|^2}{T}}^\infty e^{-\frac{s}{2}} \frac{ds}{2\pi s} \\ &\leqslant \int_{|u-y|^2/T}^\infty e^{-s/2} \frac{ds}{2\pi} \leqslant C e^{-|u-y|^2/2T}\,.\end{aligned}$$

It is thus enough to show that a.s. in X:

$$\sup_{n \geqslant 1} \sup_{y \in B(0,R)} \int_{\mathbb{R}^2} e^{-\frac{|u-y|^2}{2T}} M_n(du) < \infty\,. \tag{2.26}$$

For this, note that the term

$$\Big(\int_{\mathbb{R}^2} e^{-\frac{|u-y|^2}{2T}} M_n(du) \Big)_{n \geqslant 1}$$

is a uniformly integrable martingale. Indeed, we have:

$$\begin{aligned}&\int_{\mathbb{R}^2} e^{-\frac{|u-y|^2}{2T}} M_n(du) \\ &\leqslant \sum_{l,k \in \mathbb{Z}} e^{-\frac{(|(l,k)| - R - \sqrt{T})_+^2}{2T}} M_n([l, l+1[\times [k, k+1[)\end{aligned}$$

If $p \in]1, \frac{4}{\gamma^2}[$), by using the convexity of $x \to x^p$ and the stationarity of the measure M_n, we deduce the existence of some constant $C > 0$ (depending on R, T) such that:

$$\mathbb{E}\left[\left(\int_{\mathbb{R}^2} e^{-\frac{|u-y|^2}{2T}} M_n(du)\right)^p\right] \leqslant C\mathbb{E}[M_n([0,1[^2)^p]$$

Therefore, we get the desired result since $\sup_{n \geqslant 1} \mathbb{E}[M_n([0,1[^2)^p] < \infty$. □

Proof of Lemma 2.19. Following the above proof, it is straightforward to check that as $T \to 0$, the term corresponding to $|u-y| \leqslant \sqrt{T}$ a.s. goes to zero. For the second term corresponding to $|u-y| > \sqrt{T}$, let us rewrite it as follows (since we are interested in $T \to 0$, we may assume $T < 1/4$): we sum over annuli around y of radii $2^k\sqrt{T}$, $k = 0, \ldots, \log_2(1/\sqrt{T})$ and we also use that if $z \geqslant 1$, then for any $T < 1/4$, $e^{-z^2/2T} \leqslant T\, e^{-z^2/4T}$:

$$\begin{aligned}\int_{\mathbb{R}^2} e^{-\frac{|u-y|^2}{2T}} M_n(du) &\leqslant \sum_{k=0}^{\log_2(1/\sqrt{T})} e^{-2^{2k-3}} M_n(y, 2^k\sqrt{T}) + T \int_{\mathbb{R}^2} e^{-\frac{|u-y|^2}{4T}} M_n(du) \\ &\leqslant \widetilde{C}\, T^{\alpha/4} \sum_{k \geqslant 1} e^{-2^{2k-3}} 2^{k\alpha/2} + T \int_{\mathbb{R}^2} e^{-\frac{|u-y|^2}{4T}} M_n(du)\,, \end{aligned} \tag{2.27}$$

where, once again the constant $\widetilde{C} < \infty$ is the random constant from Theorem 2.22. One can conclude using the above fact that $\left(\int_{\mathbb{R}^2} e^{-\frac{|u-y|^2}{2T}} M_n(du)\right)_{n \geqslant 1}$ is a uniformly integrable martingale and the fact that $T \mapsto e^{-x^2/4T}$ is decreasing in T (notice also the factor T in (2.27) in front of this u.i. martingale). □

It is easy to check that the above proof of Lemma 2.17 allows us to have the following control of a log-singularity by the measure M:

Corollary 2.21. *Almost surely in X, for all $R > 0$, one has*

$$\begin{aligned}\sup_n \sup_{y \in B(0,R)} \int_{\mathbb{R}^2} \log_+ \frac{1}{|u-y|} M_n(du) &< \infty, \\ \sup_{y \in B(0,R)} \int_{\mathbb{R}^2} \log_+ \frac{1}{|u-y|} M(du) &< \infty.\end{aligned}$$

The above corollary will be instrumental in the companion paper [39].

2.8 Modulus of continuity of M

The purpose of this subsection is to prove that one can integrate the Liouville measure against **any** log-singularity. We start with a stronger result of independent interest on the *modulus of continuity* of the Liouville measure M.

Theorem 2.22. *Let $\epsilon > 0$ and $R > 0$. We set $\alpha = 2(1 - \frac{\gamma}{2})^2 > 0$. Almost surely in X, there exists a random constant $C > 0$ such that:*

$$\sup_{x \in [-R,R]^2} M(B(x,r)) \leqslant C r^{\alpha - \epsilon}, \quad \forall r \in (0,1)$$

and

$$\sup_{x \in [-R,R]^2} \sup_{n \geqslant 1} M_n(B(x,r)) \leqslant C r^{\alpha - \epsilon}, \quad \forall r \in (0,1).$$

Proof. We prove the first statement. The proof of the second one is exactly the same since $(M_n)_n$ is a martingale (use Doob's inequalities to estimate the $\sup_n M_n$ in terms of M). Recall that:

$$\xi_M(p) = (2 + \frac{\gamma^2}{2})p - \frac{\gamma^2}{2} p^2.$$

We take $R = \frac{1}{2}$ for simplicity. Now, we partition $[-\frac{1}{2}, \frac{1}{2}]^2$ into 2^{2n} dyadic squares $(I_n^j)_{1 \leqslant j \leqslant 2^{2n}}$. If p belongs to $]0, \frac{4}{\gamma^2}[$, we get:

$$\begin{aligned} P(\sup_{1 \leqslant j \leqslant 2^{2n}} M(I_n^j) \geqslant \frac{1}{2^{(\alpha-\epsilon)n}}) &\leqslant 2^{p(\alpha-\epsilon)n} \mathbb{E}[\sum_{1 \leqslant j \leqslant 2^{2n}} M(I_n^j)^p] \\ &\leqslant C_p 2^{p(\alpha-\epsilon)n} 2^{(2-\xi_M(p))n} \\ &\leqslant \frac{C_p}{2^{(\xi_M(p)-2-(\alpha-\epsilon)p)n}} \end{aligned}$$

By tacking $p = \frac{2}{\gamma}$ in the above inequalities (i.e. $\xi_M(p) - 2 - (\alpha - \epsilon)p > 0$), we get that:

$$\sup_{1 \leqslant j \leqslant 2^{2n}} M(I_n^j) \leqslant \frac{C}{2^{(\alpha-\epsilon)n}}, \quad \forall n \geqslant 1.$$

Let $r > 0$. There exists n such that $\frac{1}{2^{n+1}} < r \leqslant \frac{1}{2^n}$. We conclude by the fact that the ball $B(x,r)$ is contained in at most 4 dyadic squares in $(I_{n-1}^j)_{1 \leqslant j \leqslant 2^{2(n-1)}}$. □

2.9 Spatial continuity in law of the time-change

Now, we investigate continuity in law of the mapping $F(y, \cdot)$ with respect to the parameter y. To this purpose, we need to specify the Brownian motion that we use in the construction of the law of $(F^n(y,\cdot))_n$. Let us fix $R > 0$. For every two points $x, y \in B(0,R)$, we consider two independent Brownian motions B^x and B^y respectively starting from x and y. We couple them as prescribed by Lemma 2.15 in order to get two new Brownian motions, say $B^{x,y}$ and $B^{y,x}$ respectively starting from x and y, which coincide after some random time $\tau_{x,y}$ that goes in probability towards 0 when $|x - y| \to 0$. We use these two Brownian motions to get a coupling of $(F^n(x,\cdot))_n$ and $(F^n(y,\cdot))_n$. We claim:

Lemma 2.23. *For all $0 < s < t$ and $\eta > 0$, almost surely in X, for all $x, y \in B(0, R)$, we have*

$$\lim_{|y-x|\to 0} \limsup_{n,n'\to 0} \mathbb{P}^B\Big(\Big|F^n(y,[s,t]) - F^{n'}(x,[s,t])\Big| \geqslant \eta\Big) = 0.$$

Proof. Because the ball $B(0, R)$ is compact, we do not lose in generality if we assume that x, y converge towards the same point in this ball, say x_0. Let us fix $\epsilon > 0$. We can use the coupling procedure of Lemma 2.15 to couple simultaneously the Brownian motions $B^{x,y}$ and $B^{y,x}$ with a third independent Brownian motion, call it B, starting from a rational point $z \in \mathbb{Q}^2$: it suffices to require that the coupling occurs after $\tau_{x,y}$. Hence, we get another Brownian motion B^z, which starts from z and coincides with $B^{x,y}$ and $B^{y,x}$ after some random time τ, which satisfies $\mathbb{P}(\tau > \eta) \to 0$ (for $\eta > 0$) when $\max(|x - z|, |y - z|) \to 0$.

We can then consider the families $(F^n(x,\cdot))_n$, $(F^n(y,\cdot))_n$ and $(F^n(z,\cdot))_n$, each of which respectively constructed with the Brownian motions $B^{x,y}$, $B^{y,x}$ and B^z.

As $x, y \to x_0$, we can choose z such that $\mathbb{P}(\tau \geqslant s) \leqslant \epsilon$. We have for $\eta > 0$:

$$\begin{aligned}
&\mathbb{P}^{B^{x,y},B^{x,y},B^z}\Big(\Big|F^n(y,[s,t]) - F^{n'}(x,[s,t])\Big| \geqslant \eta\Big) \\
&\qquad\leqslant \mathbb{P}^{B^{x,y},B^{x,y},B^z}\Big(\Big|F^n(y,[s,t]) - F^{n'}(x,[s,t])\Big| \geqslant \eta, \tau \geqslant s\Big) \\
&\qquad\quad + \mathbb{P}^{B^{x,y},B^{x,y},B^z}\Big(\Big|F^n(y,[s,t]) - F^{n'}(x,[s,t])\Big| \geqslant \eta, \tau < s\Big) \\
&\qquad\leqslant \epsilon + \mathbb{P}^{B^z}\Big(\Big|F^n(z,[s,t]) - F^{n'}(z,[s,t])\Big| \geqslant \eta\Big).
\end{aligned}$$

Since $z \in \mathbb{Q}^2$, we have

$$\limsup_{n,n'\to\infty} \mathbb{P}^B\Big(\Big|F^n(z,[s,t]) - F^{n'}(z,[s,t])\Big| \geqslant \eta\Big) = 0.$$

The proof is complete. ☐

For each $y \in \mathbb{R}^2$, we consider a Brownian motion B^y starting from y and the family $(F^n(y,\cdot))_n$ constructed with the Brownian motion B^y.

Proposition 2.24. *Almost surely in X, for all $y \in \mathbb{R}^2$, the family $(B^y, F^n(y,\cdot))_n$ converges in law under $\mathbb{P}^{B^y}$ in $C(\mathbb{R}_+, \mathbb{R}^2) \times C(\mathbb{R}_+, \mathbb{R}_+)$ equipped with the sup-norm topology over compact sets towards a limiting couple $(B^y, F(y,\cdot))$ where the mapping $F(y,\cdot)$ is continuous, increasing and satisfies:*

$$\forall y \in \mathbb{R}^2, \quad \lim_{t\to\infty} F(y,t) = +\infty. \tag{2.28}$$

Furthermore, almost surely in X, the mapping

$$y \mapsto (B^y, F(y,\cdot))$$

is continuous in law in $C(\mathbb{R}_+, \mathbb{R}^2) \times C(\mathbb{R}_+, \mathbb{R}_+)$.

Proof of Proposition 2.24. By applying Theorem 2.10 on all the rational points of $\mathbb{R}^2$ together with Proposition 2.16, we prove that, almost surely in X, for all $x \in \mathbb{R}^2$, the family $(B^x, F^n(x,\cdot))_n$ is tight in law under $\mathbb{P}^B$ in $C(\mathbb{R}_+,\mathbb{R}^2)\times C(\mathbb{R}_+,\mathbb{R}_+)$ equipped with the sup-norm topology over compact sets. Furthermore, we may assume that convergence in law holds for all the rational points of $\mathbb{R}^2$. With the help of Lemma 2.23, we prove that, for each given point $x\in\mathbb{R}^2$, there is a unique possible limiting law, which is characterized by the laws of $(B^x, F(x,\cdot))_{x\in\mathbb{Q}^2}$. Therefore, for each $x \in \mathbb{R}^2$, the family $(B^x, F(x,\cdot))_n$ converges in law under $\mathbb{P}^{B^x}$ in $C(\mathbb{R}_+,\mathbb{R}^2)\times C(\mathbb{R}_+,\mathbb{R}_+)$ equipped with the sup-norm topology towards a limiting couple $(B^x, F(x,\cdot))$, where B^x is a standard Brownian motion and the mapping $F(x,\cdot)$ is continuous and nondecreasing in its input t. Let us denote by $\mathbb{P}^x$ the law of $(B^x, F(x,\cdot))$ in $C(\mathbb{R}_+,\mathbb{R}^2)\times C(\mathbb{R}_+,\mathbb{R}_+)$.

We deduce from Lemma 2.23 that, when x converges to x_0, the random measure $F(x,dr)$ (together with B^x) converges in law under $\mathbb{P}^x$ in the sense of weak convergence of measures towards the law of the random measure $F(x_0,dr)$ under $\mathbb{P}^{x_0}$ (together with B^{x_0}). From this and Lemma 2.25 below, we conclude that the law of $(B^x, F(x,\cdot))$ under $\mathbb{P}^x$ converges in $C(\mathbb{R}_+,\mathbb{R}^2)\times C(\mathbb{R}_+,\mathbb{R}_+)$ towards the law of $(B^{x_0}, F(x_0,\cdot))$ under $\mathbb{P}^{x_0}$ when $x\to x_0$.

All what remains to prove is that for all y, the mapping $t\mapsto F(y,[0,t])$ is increasing under $\mathbb{P}^y$. From Theorem 2.10, we may assume that, almost surely in X, for all the rational points $y\in\mathbb{R}^2$, the mapping $t\mapsto F(y,[0,t])$ is a.s. increasing under $\mathbb{P}^y$. So, let us fix $y_0\in\mathbb{R}^2$ and let us prove that the mapping $t\mapsto F(y_0,[0,t])$ is a.s. increasing under $\mathbb{P}^{y_0}$. It is enough to prove that $\mathbb{P}^{y_0}(F(y_0,[s,t])>0)$ for a countable family of intervals $[s,t]$ generating the Borel topology on $\mathbb{R}_+$. Let us consider such an interval $[s,t]$ with $s>0$. We will use the coupling argument of Lemma 2.15: let us denote the corresponding Brownian motion $\overline{B}^{y,y_0}$. Of course, the law $\mathbb{P}^{y_0}$ does not depend on the Brownian motion that we choose to define F^n. So we are free to choose the Brownian motion $\overline{B}^{y,y_0}$. Then we have

$$\begin{aligned}\mathbb{P}^{y_0}(F(y_0,[s,t])>0) &\geqslant \mathbb{P}^{y_0}(F(y_0,[s,t])>0, \tau_2^y<s)\\ &= \mathbb{P}^{y}(F(y,[s,t])>0, \tau_2^y<s)\\ &\geqslant \mathbb{P}^{y}(F(y,[s,t])>0) - \mathbb{P}(\tau_2^y\geqslant s)\\ &\geqslant 1-\mathbb{P}^y(\tau_2^y\geqslant s).\end{aligned}$$

Clearly, $\mathbb{P}^y(\tau_2^y\geqslant s)\to 0$ as $y\to y_0$ in such a way that $\mathbb{P}^{y_0}(F(y_0,[s,t])>0)=1$.

This coupling argument also applies to deduce from Theorem 2.10 that, a.s. in X, for all $y\in\mathbb{R}^2$, $\mathbb{P}^y$ a.s.:

$$\lim_{t\to\infty} F(y,t)=+\infty.$$

□

Lemma 2.25. *Let us denote by M_T the set of finite Radon measures on $[0,T]$ equipped with the topology of weak convergence of measures. Let $(\mu_n)_n$ be a sequence of random elements in M_T converging in law towards μ. Assume that μ and each μ_n*

is diffuse. Then the random mapping $t \mapsto \mu_n([0,t])$ converges in law in $C([0,T])$ as $n \to \infty$ towards the random mapping $t \mapsto \mu([0,t])$.

Proof. It is well-known that the topology of weak-convergence for measures on M_T is metrizable, the so-called Prohorov's metric being one of the possible choices. Recall that the Prohorov metric on the space M_T is defined as follows (see for example [62]). For any $\mu, \nu \in M_T$, let

$$d_{M_T}(\mu,\nu) := \inf \left\{ \varepsilon > 0\,, \text{ s.t } \forall \text{ closed set } A \subset [0,T]\,, \begin{array}{c} \mu(A^\varepsilon) \leq \nu(A) + \varepsilon \\ \text{and} \\ \nu(A^\varepsilon) \leq \mu(A) + \varepsilon \end{array} \right\}, \tag{2.29}$$

where A^ε-denotes the ε-neighborhood of A.

It is well-known (see [62]) that the metric space (M_T, d_{M_T}) is a complete separable metric space. In particular, one can apply Skorohod representation theorem. With a slight abuse of notation, we may thus couple μ_n and μ on the same probability space so that a.s. $d_{M_T}(\mu_n,\mu) \to 0$.

Let us define,

$$F_n(t) = \mu_n([0,t]), \quad F(t) = \mu([0,t]).$$

Since $[0,T]$ is compact and since μ is assumed to be diffuse, we have that F is a.s. uniformly continuous. Let δ_F be its modulus of continuity. For any fixed $\alpha > 0$, let $\varepsilon > 0$ be such that $\delta_F(2\varepsilon) < \alpha$ and N large enough so that for each $n \geqslant N$, $d_{M_T}(\mu_n,\mu) < \varepsilon$. As such we have for all $n \geqslant N$ and for all $t \in [0,T]$,

$$F(t-2\varepsilon) - 2\varepsilon \leqslant F_n(t) \leqslant F(t+2\varepsilon) + 2\varepsilon\,,$$

which shows that $\|F - F_n\|_\infty < \alpha$, for all $n \geqslant N$. Since α was arbitrary we thus showed that a.s. $\|F_n - F\|_\infty \to 0$. ☐

2.10 Defining the Liouville Brownian motion

By fixing a given point $x \in \mathbb{R}^2$, we are now able to define the LBM starting from x almost surely w.r.t. the field X and the Brownian motion B:

Theorem 2.26. *Assume $\gamma^2 < 4$ and fix $x \in \mathbb{R}^2$. Almost surely in X and in B, the n-regularized Brownian motion $(\mathcal{B}^{n,x})_n$ defined by Definition 2.2 converges in the space $C(\mathbb{R}_+,\mathbb{R}^2)$ equipped with the supremum norm on compact sets towards a continuous random process $\mathcal{B}^x$, which we call (massive)* **Liouville Brownian motion** *starting from x, characterized by:*

$$\mathcal{B}^x_{\mathbf{t}} = x + B_{\langle \mathcal{B}^x \rangle_{\mathbf{t}}}$$

where $\langle \mathcal{B}^x \rangle$ is defined by

$$F(x, \langle \mathcal{B}^x \rangle_{\mathbf{t}}) = \mathbf{t}.$$

Furthermore, $\mathcal{B}^x$ is a local martingale.

As a consequence, almost surely in X, the n-regularized Liouville Brownian motion defined in Definition 2.1 converges in law under $\mathbb{P}^B$ in $C(\mathbb{R}_+,\mathbb{R}^2)$ towards $\mathcal{B}^x$.

Proof of Theorem 2.26. It is just a consequence of Corollary 2.11. □

This result will allow us to prove that, almost surely in X, we can define the law of the Liouville Brownian motion for all possible starting point $y \in \mathbb{R}^2$:

Theorem 2.27. *Assume $\gamma^2 < 4$. Almost surely in X, for all $y \in \mathbb{R}^2$, the n-regularized Brownian motion $(\mathcal{B}^{n,y})_n$ defined by Definition 2.2 converges in law in the space $C(\mathbb{R}_+, \mathbb{R}^2)$ towards a continuous random process $\mathcal{B}^y$, characterized by:*

$$\mathcal{B}^y_{\mathbf{t}} = y + B_{\langle \mathcal{B}^y \rangle_{\mathbf{t}}}$$

where $\langle \mathcal{B}^y \rangle$ is defined by

$$F(y, \langle \mathcal{B}^y \rangle_{\mathbf{t}}) = \mathbf{t}.$$

Furthermore, almost surely in X and under $\mathbb{P}^B$, the law of the mapping $y \mapsto \mathcal{B}^y$ is continuous in law in $C(\mathbb{R}_+, \mathbb{R}^2)$.

Proof of Theorem 2.27. All the statements of Theorem 2.27 are a direct consequence of Proposition 2.24 excepted the continuity in law in $C(\mathbb{R}_+, \mathbb{R}^2)$ of the quadratic variations. Let us prove this statement. Rigorously, we should prove the continuity in law of the couple $(B^x, F(x, \cdot)^{-1})$ with respect to the parameter x. With a slight abuse, we will omit the first component B^x in the following argument to simplify the notations (and therefore only focus on the continuity of $F(x, \cdot)^{-1}$). Nevertheless, it is clear that the argument works for the couple $(B^x, F(x, \cdot)^{-1})$. Fix $n \in \mathbb{N}$ and $0 \leqslant t_1 < \cdots < t_n$ and consider the mapping

$$\varphi \in C(\mathbb{R}_+, \mathbb{R}_+) \mapsto (\varphi^{-1}(t_1), \dots, \varphi^{-1}(t_n))$$

where

$$\varphi^{-1}(t) = \inf\{s \geqslant 0; \varphi(s) > t\} \quad \text{with } \inf \emptyset = +\infty.$$

It is well defined for all functions $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$. Observe that it is continuous at all those functions that are continuous increasing and going to ∞ as $t \to \infty$. Since $\mathbb{P}^x$ gives full measures to those functions, we deduce that, almost surely in X, $(F^{-1}(y, t_1), \dots, F^{-1}(y, t_n))$ converges in law under $\mathbb{P}^y$ towards $(F^{-1}(x, t_1), \dots, F^{-1}(x, t_n))$ as $y \to x$.

We can apply this result for all the dyadic points $t \in \mathbb{R}_+$. We deduce that, almost surely in X, for all $x \in \mathbb{R}^2$, the random measure $F^{-1}(y, dr)$ converges in law under $\mathbb{P}^y$ in the space of Radon measures on $\mathbb{R}_+$ towards $F^{-1}(x, dr)$ under $\mathbb{P}^x$ as $y \to x$. By using Lemma 2.25 once again, we deduce that $F^{-1}(y, \cdot)$ converges in law under $\mathbb{P}^y$ in $C(\mathbb{R}_+)$ towards $F^{-1}(x, \cdot)$ under $\mathbb{P}^x$ as $y \to x$. □

Also, by noticing that the Liouville Brownian motion is a time change of a standard planar Brownian motion with quadratic variations going to infinity, it is plain to deduce that:

Corollary 2.28. *Almost surely in X, the Liouville Brownian motion is recurrent, i.e. for all $x \in \mathbb{R}^2$ and for all $z \in \mathbb{R}^2$:*

$$\mathbb{P}^x\big[\liminf_{t\to\infty} |\mathcal{B}_t - z| = 0\big] = 1.$$

Furthermore, almost surely in X, for all $x \in \mathbb{R}^2$,

$$\mathbb{P}^x\big[\limsup_{t\to\infty} |\mathcal{B}_t| = \infty\big] = 1.$$

Remark 2.29. *The whole convergence of the quadratic variations in $C(\mathbb{R}_+)$ allows to deal with quite general a family of functionals of this process. As a straightforward consequence, we will see in section 4 that the semi-group of the n-regularized LBM converges towards the semi-group of the LBM. For instance, we can also deal with exit times of smooth enough domains (say satisfying a Zaremba's cone condition for instance), giving a probabilistic interpretation to PDEs involving the Liouville Laplacian (see below) with boundary conditions of Dirichlet type. For instance, one can give rigorous a meaning to equations of the type:*

$$\triangle u = \mu\, e^{\gamma X(x) - \frac{\gamma^2}{2}\mathbb{E}[X(x)^2]} \quad \text{for } x \in D, \quad u(x) = f(x) \quad \text{for } x \in \partial D.$$

We let the reader think of all the other possible uses of this convergence.

2.11 Markov and Feller property

The aim of this subsection is to prove the following theorem.

Theorem 2.30. *The Liouville Brownian motion is a Markov process. Furthermore, it is a Feller process.*

Proof of Theorem 2.30. We start by proving the Markov property. Let us fix $\mathbf{s} < \mathbf{t}$ and $x \in \mathbb{R}^2$. Recall that we have the following expression:

$$\langle \mathcal{B}^{n,x}\rangle_{\mathbf{t}} - \langle \mathcal{B}^{n,x}\rangle_{\mathbf{s}} = \inf\{v \geqslant 0, F^n(\mathcal{B}^{n,x}_{\mathbf{s}}, v) \geqslant \mathbf{t} - \mathbf{s}\}$$

where F^n is constructed with the Brownian motion $\bar{B}_v = B_{v+\langle \mathcal{B}^{n,x}\rangle_{\mathbf{s}}} - B_{\langle \mathcal{B}^{n,x}\rangle_{\mathbf{s}}}$, which is independent from $(B_u)_{u \leqslant \langle \mathcal{B}^{n,x}\rangle_{\mathbf{s}}}$.

Let us consider two continuous bounded functionals H, G on $C(\mathbb{R}_+, \mathbb{R}^2)$. By applying the Markov property to the standard Brownian motion, we get:

$$\begin{aligned}
&\mathbb{E}^x[H((B_u)_{u \leqslant \langle \mathcal{B}^{n,x}\rangle_{\mathbf{s}}}) G((B_u)_{u \geqslant \langle \mathcal{B}^{n,x}\rangle_{\mathbf{s}}}, F^n(\mathcal{B}^{n,x}_{\mathbf{s}}, v)_{v\geqslant 0})]\\
&= \mathbb{E}^x[H((B_u)_{u \leqslant \langle \mathcal{B}^{n,x}\rangle_{\mathbf{s}}}) \mathbb{E}^{\mathcal{B}^{n,x}_{\mathbf{s}}}[G((\mathcal{B}^{n,x}_{\mathbf{s}} + \bar{B}_v)_{v\geqslant 0}, F^n(\mathcal{B}^{n,x}_{\mathbf{s}}, v)_{v\geqslant 0})]]
\end{aligned}$$

where $\bar{B}$ is a Brownian motion starting from $\mathcal{B}^{n,x}_{\mathbf{s}}$. Following the argument of Theorem 2.27, the following property holds: for all $y \in \mathbb{R}^2$ and all sequence $(y_n)_{n\geqslant 1}$

going to y, the law of the triplet $(B, F^n(y_n,\cdot), \langle \mathcal{B}^{n,y_n}\rangle)$ under $\mathbb{P}^{y_n}$ converges to that of the triplet $(B, F(x,\cdot), \langle \mathcal{B}^x\rangle)$ under $\mathbb{P}^x$. We can then take the limit in the above identity to show:

$$
\begin{aligned}
&\mathbb{E}^x[H((B_u)_{u\leqslant \langle \mathcal{B}^x\rangle_{\mathbf{s}}})G((B_u)_{u\geqslant \langle \mathcal{B}^x\rangle_{\mathbf{s}}}, F(\mathcal{B}^x_{\mathbf{s}}, v)_{v\geqslant 0})] \\
&= \mathbb{E}^x[H((B_u)_{u\leqslant \langle \mathcal{B}^x\rangle_{\mathbf{s}}})\mathbb{E}^{\mathcal{B}^x_{\mathbf{s}}}[G((\mathcal{B}^x_{\mathbf{s}} + \bar{B}_v)_{v\geqslant 0}, F(\mathcal{B}^x_{\mathbf{s}}, v)_{v\geqslant 0})]].
\end{aligned}
$$

The Markov property is then an immediate consequence of the above identity since $\mathcal{B}^x_{\mathbf{t}} = \mathcal{B}^x_{\mathbf{s}} + \bar{B}_{\langle \mathcal{B}^{n,x}\rangle_{\mathbf{t}} - \langle \mathcal{B}^{n,x}\rangle_{\mathbf{s}}}$.

We now turn to the Feller property. Let us consider a continuous bounded function f. Fix $x\in\mathbb{R}^2$. Since the mapping $y\mapsto \mathcal{B}^y$ is continuous in law in $C(\mathbb{R}_+)$, we deduce that $\mathcal{B}^y_{\mathbf{t}}$ converges in law towards $\mathcal{B}^x_{\mathbf{t}}$ as $y\to x$. Therefore the mapping $y\mapsto \mathbb{E}^y[f(\mathcal{B}^y_{\mathbf{t}})]$ is continuous at x. □

It is then a routine trick to deduce

Corollary 2.31. *The Liouville Brownian motion is a strong Markov process.*

2.12 Asymptotic independence of the Liouville Brownian motion and the Euclidean Brownian motion

In this subsection, we make rigorous the statement in Remark 2.3. Recall that the Euclidean Brownian motion $\bar{B}$ is the one involved in Definition 2.1.

Theorem 2.32. *If $\gamma < 2$, almost surely in X, the couple of processes $(\bar{B}, \mathcal{B}^n)_n$ converges in law towards a couple $(\bar{B}, \mathcal{B})$. The Euclidean Brownian motion $\bar{B}$ and the Liouville Brownian motion $\mathcal{B}$ (or equivalently B) are independent.*

Surprisingly, the above theorem shows that some extra-randomness is created by taking the limit $n\to\infty$. Indeed, the n-regularized Liouville Brownian motion is a measurable function of the Euclidean Brownian motion. Yet, Liouville/Euclidean Brownian motions are independent at the limit.

As a consequence the Liouville Brownian motion, as defined in (2.5) cannot converge in a stronger sense than in law. This justifies our approach of studying the convergence via the Dambis-Schwarz representation theorem.

Proof of Theorem 2.32. Before beginning the proof, let us first clarify a few points. The n-regularization of the Liouville Brownian motion (2.5) involves the Euclidean Brownian motion $\bar{B}$. An equivalent definition of this n-regularization is given in Definition 2.2 by means of another Brownian motion B, constructed via the Dambis-Schwarz theorem. As such, it implicitly depends on n as well as $\bar{B}$. It is therefore relevant to write explicitly this dependence in this proof. So we will write B^n instead of B. It turns out that, as $n\to\infty$, the proofs of Theorems 2.26 and 2.27 show that the Liouville Brownian motion is a measurable function of the Brownian motion B. The frame of our proof will thus be the following. First, we write explicitly the

dependence structure between $\bar{B}$ and B^n. Second, we prove that, at the limit $n \to \infty$, they are independent.

So, as announced, we begin with writing explicitly the dependence between $\bar{B}$ and B^n. The Dambis-Schwarz theorem tells us that

$$x + B_t^{n,x} = \mathcal{B}^n_{\tau_t^{n,x}}$$

where

$$\tau_t^{n,x} = \inf\{s \geqslant 0; \langle \mathcal{B}^{n,x}\rangle_s > t\}.$$

From (2.9) and Lemma 2.7, we deduce:

$$\tau_t^{n,x} = F^n(x,t).$$

Therefore

$$B_t^{n,x} = \int_0^{F^n(x,t)} e^{-\frac{\gamma}{2}X_n(\mathcal{B}^{n,x}_{\mathbf{u}})+\frac{\gamma^2}{4}\mathbb{E}[X_n(\mathcal{B}^{n,x}_{\mathbf{u}})^2]}\, d\bar{B}_{\mathbf{u}} = \mathcal{B}^{n,x}_{F^n(t,x)}.$$

Now we prove asymptotic independence of $\mathcal{B}$ and $\bar{B}$. Let us compute their predictable bracket:

$$\begin{aligned}\langle \mathcal{B}^{n,x}, \bar{B}\rangle_t &= \int_0^t e^{-\frac{\gamma}{2}X_n(\mathcal{B}^{n,x}_r)+\frac{\gamma^2}{4}\mathbb{E}[X_n(\mathcal{B}^{n,x}_r)^2]}\, dr \\ &= \int_0^t e^{-\frac{\gamma}{2}X_n(x+B_{\langle \mathcal{B}^{n,x}\rangle_r})+\frac{\gamma^2}{4}\mathbb{E}[X_n(x+B_{\langle \mathcal{B}^{n,x}\rangle_r})^2]}\, dr.\end{aligned}$$

By making the change of variables

$$u = \langle \mathcal{B}^{n,x}\rangle_r, \quad e^{\gamma X_n(x+B_u)-\frac{\gamma^2}{2}\mathbb{E}[X_n(x+B_u)^2]}\, du = dr,$$

we get:

$$\begin{aligned}\langle \mathcal{B}^{n,x}, \bar{B}\rangle_t &= \int_0^{\langle \mathcal{B}^{n,x}\rangle_t} e^{\frac{\gamma}{2}X_n(x+B_u)-\frac{\gamma^2}{4}\mathbb{E}[X_n(x+B_u)^2]}\, du \\ &= e^{-\frac{\gamma^2}{8}\mathbb{E}[X_n(x+B_u)^2]} \int_0^{\langle \mathcal{B}^{n,x}\rangle_t} e^{\frac{\gamma}{2}X_n(x+B_u)-\frac{\gamma^2}{8}\mathbb{E}[X_n(x+B_u)^2]}\, du.\end{aligned}$$

Let us prove that this latter quantity converges in probability towards 0 when t is fixed. Theorem 2.10 implies that, almost surely in X, the mapping

$$t \mapsto \int_0^t e^{\frac{\gamma}{2}X_n(x+B_u)-\frac{\gamma^2}{8}\mathbb{E}[X_n(x+B_u)^2]}\, dr$$

converges in law in $C(\mathbb{R}_+)$. Therefore

$$e^{-\frac{\gamma^2}{8}\mathbb{E}[X_n(x+B_u)^2]} \int_0^{\langle \mathcal{B}^{n,x}\rangle_t} e^{\frac{\gamma}{2}X_n(x+B_u)-\frac{\gamma^2}{8}\mathbb{E}[X_n(x+B_u)^2]}\, du$$

converges in law in $C(\mathbb{R}_+)$ towards 0 since $e^{-\frac{\gamma^2}{8}\mathbb{E}[X_n(x+B_u)^2]} \to 0$ as $n \to \infty$ (this quantity is independent of x, u, and $\mathrm{Var}(X_n) \to \infty$ as $n \to \infty$). Furthermore, for t fixed,

$$\mathbb{P}^{\bar{B}}(\langle \mathcal{B}^{n,x} \rangle_t > R) \to 0, \quad \text{uniformly w.r.t } n.$$

It is plain to deduce that, almost surely in X, under $\mathbb{P}^{\bar{B}}$ the sequence $(\langle \mathcal{B}^{n,x}, \bar{B} \rangle_t)_n$ converges in law towards 0 as $n \to \infty$ and therefore in $\mathbb{P}^{\bar{B}}$-probability. Since the mapping $t \mapsto \langle \mathcal{B}^{n,x}, \bar{B} \rangle_t$ is nondecreasing, we deduce that in $\mathbb{P}^{\bar{B}}$-probability, the process $t \mapsto \langle \mathcal{B}^{n,x}, \bar{B} \rangle_t$ converges towards 0 in $C(\mathbb{R}_+)$. Knight's theorem [47, Theorem 4.13] implies that B and $\bar{B}$ are independent (See also the appendix in [61]). As a measurable function of B, the Liouville Brownian motion is independent of $\bar{B}$. □

2.13 Remarks about associated Feynman path integrals

Feynman path integrals have been introduced in order to produce probability measures on curves the energy of which are expressed in terms of Lagrangians instead of Hamiltonians. Remind that the standard Wiener measure gives a rigorous interpretation of the heuristic path integral on $\mathbb{R}^2$

$$\frac{1}{Z_0}\int_{C([0,T];\mathbb{R}^2)} f(\sigma)\exp\Big(-\frac{1}{2}\int_0^T |\sigma'(s)|^2\,ds\Big)\mathcal{D}\sigma \tag{2.30}$$

where Z_0 appears as a normalization constant and $f : C([0,T],\mathbb{R}^2) \to \mathbb{R}$ is a bounded continuous function. It turns out that the construction of the Liouville Brownian motion allows us to make sense of several Feynman path integrals appearing in Liouville quantum gravity literature. We discuss below these integrals.

The "Wiener measure" associated to the Liouville Brownian motion has the following path integral interpretation:

$$\mathbb{E}^B\big[f\big((\mathcal{B}_t)_{0\leqslant t\leqslant T}\big)\big] = \frac{1}{Z_1}\int_{C([0,T];\mathbb{R}^2)} f(\sigma)\exp\Big(-\frac{1}{2}\int_0^T e^{\gamma X(\sigma(s))-\frac{\gamma^2}{2}\mathbb{E}[X^2]}|\sigma'(s)|^2\,ds\Big)\mathcal{D}\sigma,$$

where Z_1 is a normalization constant, valid for all bounded continuous function $f : C([0,T],\mathbb{R}^2) \to \mathbb{R}$.

We can also give a rigorous meaning to the following path integral on $\mathbb{R}^2$:

$$\begin{aligned}\frac{1}{Z_0}\int_{C([0,T];\mathbb{R}^2)} f(\sigma)\exp\Big(-\frac{1}{2}\int_0^T |\sigma'(s)|^2 + \mu e^{\gamma X(\sigma(s))-\frac{\gamma^2}{2}\mathbb{E}[X^2]}\,ds\Big)\mathcal{D}\sigma \\ = \mathbb{E}^B\Big[f\big((B_t)_{0\leqslant t\leqslant T}\big)e^{-\mu\int_0^T e^{\gamma X(B_r)-\frac{\gamma^2}{2}\mathbb{E}[X^2]}\,dr}\Big].\end{aligned} \tag{2.31}$$

Of course, Z_0 is the renormalization constant of the Wiener measure so that, as written in (2.31), the path integral is not normalized. We can renormalize it by replacing Z_0 by

$$Z_\mu = \mathbb{E}^B\Big[e^{-\mu\int_0^T e^{\gamma X(B_r)-\frac{\gamma^2}{2}\mathbb{E}[X^2]}\,dr}\Big].$$

Further comments can be made if we further assume that f is nonnegative.

Proposition 2.33. *Assume that f is nonnegative. Then the Feynman path integral*

$$\frac{1}{Z_0}\int_{C([0,T];\mathbb{R}^2)} f(\sigma)\exp\Big(-\frac{1}{2}\int_0^T |\sigma'(s)|^2+\mu e^{\gamma X(\sigma(s))-\frac{\gamma^2}{2}\mathbb{E}[X^2]}\,ds\Big)\mathcal{D}\sigma$$

is a continuous non-increasing function of μ. Expectation of this path integral with respect to the field X is a non-decreasing function of γ.

Proof. The claimed properties with respect to the parameter μ just results from standard theorems of parameterized integrals. Because the mapping $x\mapsto e^{-\mu x}$ is convex, Kahane's convexity inequalities entails that expectation with respect to the field X of this path integral yields a non-decreasing function of γ. $\square$

3 Liouville Brownian motion defined on other geometries: torus, sphere and planar domains

So far, we constructed in detail the Liouville Brownian motion for the (Massive) Free Field on $\mathbb{R}^2$. In this section, we wish to briefly discuss how one can extend this construction to the following cases:

1. Liouville Brownian motion on the sphere $\mathbb{S}^2$ equipped with a standard Gaussian Free Field (GFF) with vanishing average.
2. Liouville Brownian motion on the torus $\mathbb{T}^2$ equipped with a GFF with vanishing average.
3. Liouville Brownian motion on a domain D (i.e. a simply connected domain $D\subsetneqq\mathbb{C}$), equipped with a GFF with Dirichlet boundary conditions.

We will not detail the proofs since the whole machinery works the same as in the plane, especially in the first two cases where the field is stationary with respect to the canonical shift of the manifold as the MFF was in $\mathbb{R}^2$. In the third case, we will briefly explain two ways to build a LBM on the domain D: either by adapting the machinery to a non-stationary GFF or by relying on an appropriate coupling argument which avoids any additional technicality. As such, we will essentially focus on formulating precisely the respective frameworks.

3.1 Liouville Brownian motion on the sphere

3.1.1 Gaussian Free Field on the sphere

We consider a Gaussian Free Field (GFF for short) on the sphere $\mathbb{S}^2$ with vanishing average. It is a standard Gaussian in the Hilbert space defined as the closure of Schwartz functions with vanishing integral over $\mathbb{S}^2$ with respect to the inner product

$$(f,g)_h=-(f,\triangle g)_2,$$

where $(\cdot,\cdot)_2$ is the standard inner product on $L^2(\mathbb{S}^2)$. Its action on $L^2(\mathbb{S}^2)$ can be seen as a Gaussian distribution with covariance kernel given by the Green function G of the operator $-\triangle$ with vanishing mean (times the normalization factor 2π).

Let us consider an orthonormal basis $(e_n)_{n\geqslant 0}$ of $L^2(\mathbb{S}^2)$ made up of eigenfunctions of the operator $-\triangle$ (precisely, the spherical harmonics). We assume that e_0 is the constant function. Let $(\lambda_n)_{n\geqslant 1}$ be the associated sequence of (positive) eigenvalues. The GFF on $\mathbb{S}^2$ is the Gaussian distribution defined by

$$X(x)=\sqrt{2\pi}\sum_{k\geqslant 1}\lambda_k^{-1/2}e_k(x)\alpha_k$$

where $(\alpha_k)_k$ is a sequence of i.i.d. standard Gaussian random variables. In that case, we define the n-regularized smooth Gaussian field

$$X_n(x)=\sqrt{2\pi}\sum_{k=1}^{n}\lambda_k^{-1/2}e_k(x)\alpha_k. \tag{3.1}$$

We stress that the correlations of the GFF on the sphere behaves at short scales like the logarithm of the Riemannian distance

$$\mathbb{E}[X(x)X(y)]=\ln_+\frac{1}{d(x,y)}+g(x,y)$$

where g is some bounded continuous function on the sphere and d is the distance induced by the Riemannian metric of the sphere. Therefore, Kahane's theory [44] applies for the GFF on the sphere.

3.1.2 Brownian motion on the sphere

Consider the unit sphere $\mathbb{S}^2$ as a submanifold of $\mathbb{R}^3$. Using classical terminology, let us denote by $\mathbf{T}\mathbb{S}^2=\bigcup_{x\in\mathbb{S}^2}\mathbf{T}_x\mathbb{S}^2$ the tangent bundle of the sphere. The Laplace-Beltrami operator on the sphere, here denoted by $\triangle$, can be written in the form of a sum of squares:

$$\triangle=\sum_{i=1}^{3}P_i^2$$

where P_i is the projection of the i-th coordinate unit vector e_i on the tangent space $\mathbf{T}_x\mathbb{S}^2$. Each P_i is a vector field on $\mathbb{S}^2$. The projection to the tangent sphere at x is given by

$$P(x)\xi=\xi-\langle\xi,x\rangle x,\quad x\in\mathbb{S}^2,\ \ \xi\in\mathbb{R}^3,$$

in such a way that the matrix $P=\{P_1,P_2,P_3\}$ can be explicitly written as:

$$P(x)_{ij}=\delta_{ij}-x_ix_j.$$

Consider the following Stratonovich stochastic differential on $\mathbb{S}^2$ driven by a 3-dimensional Euclidean Brownian motion W:

$$dB_t = \sum_{i=1}^{3} P_i(B_t) \circ dW_t^i, \quad X_0 \in \mathbb{S}^2.$$

This is a stochastic differential equation on $\mathbb{S}^2$ because P_i are vector fields on $\mathbb{S}^2$. Extending P_i arbitrarily to the whole ambient space, we can solve this equation as if it is an equation on $\mathbb{R}^3$. It can be checked that if the initial condition lies on the manifold $\mathbb{S}^2$, then the solution B lies on $\mathbb{S}^2$ for all times. Furthermore, it is a diffusion process generated by $\frac{1}{2}\triangle$.

Therefore, Brownian motion on $\mathbb{S}^2$ is the solution of the stochastic differential equation

$$B_t^i = B_0^i + \int_0^t (\delta_{ij} - B_s^i B_s^j) \circ dW_s^j, \quad B_0 \in \mathbb{S}^2.$$

This is Stroock's representation of spherical Brownian motion.

3.1.3 Construction of the Liouville Brownian motion on the sphere

The construction of the n-regularized Brownian motion on the sphere is quite similar to the standard spherical Brownian motion. It is the solution of the following stochastic differential equation:

$$\mathcal{B}_t^{n,x,i} = \mathcal{B}_0^{n,x,i} + \int_0^t (\delta_{ij} - \mathcal{B}_s^{n,x,i}\mathcal{B}_s^{n,x,j}) e^{-\frac{\gamma}{2}X_n(\mathcal{B}_s^{n,x}) + \frac{\gamma^2}{4}\mathbb{E}[X_n(\mathcal{B}_s^{n,x})]} \circ dW_s^j, \quad \mathcal{B}_0^{n,x} \in \mathbb{S}^2.$$

All the results stated in section 2 apply since Kahane's theory remains valid on C^1-manifolds (see [44]). Intuitively, this is just because such manifolds are locally isometric to open domains of the Euclidean space. In particular, curvature does not play a fundamental part.

3.2 Liouville Brownian motion on the torus $\mathbb{T}^2$

The standard GFF on $\mathbb{T}^2$ with vanishing average is defined exactly in the same fashion as on the sphere. Namely, let $(e_n)_{n \geqslant 0}$ be an orthonormal basis of $L^2(\mathbb{T}^2)$ made up of eigenfunctions of $-\triangle$ with eigenvalues $(\lambda_n)_{n \geqslant 0}$ (with e_0 constant). One could be more explicit here about the expression of the $(e_n)_{n \geqslant 0}$ but this will not be needed. The GFF on $\mathbb{T}^2$ with vanishing average is defined as well by

$$X(x) = \sqrt{2\pi} \sum_{k \geqslant 1} \lambda_k^{-1/2} e_k(x) \alpha_k \,,$$

where $(\alpha_k)_k$ is a sequence of i.i.d. standard Gaussian random variables. Exactly as in the case of the sphere, one can define a Liouville Brownian motion $(\mathcal{B}_{\mathbf{t}})_{\mathbf{t} \geqslant 0}$ on $\mathbb{T}^2$ (furthermore, there no curvature effect here).

Remark 3.1. *There is at least one point in our proofs that must be changed in order to apply to the torus or the sphere, or any bounded manifold without boundary: the fact that* $\lim_{t\to\infty} F(x,t) = +\infty$*. Indeed, our proof uses the "infinite volume" of the plane. In the case of the torus or sphere, the strategy is much simpler because of compactness arguments: the standard Brownian motion on* $\mathbb{S}^2$ *or* $\mathbb{T}^2$ *possesses an invariant probability measure, call it* μ*, which is nothing but the volume form of* $\mathbb{S}^2$ *or* $\mathbb{T}^2$*. Apply the ergodic theorem to prove that* $\mathbb{P}^X \otimes \mathbb{P}^\mu$ *almost surely:*

$$\lim_{t\to\infty} \frac{F(x,t)}{t} = G,$$

for some random variable G*, which is shift-invariant. Since the Brownian motion on the sphere is ergodic,* G *is measurable with respect to the sigma algebra generated by* $\sigma\{X_x; x \in \mathbb{T}^2 \text{ or } \mathbb{S}^2\}$*. It is not clear that* G *is constant. Yet, the set* $\{G > 0\}$ *is measurable with respect to the asymptotic sigma-algebra of the* $(X_{n+1} - X_n)_n$*. Therefore,* $\mathbb{P}^\mu$ *almost surely, the set* $\{G > 0\}$ *has* $\mathbb{P}^X$*-probability* 0 *or* 1*. Since* G *has expectation* 1*, this set has* $\mathbb{P}^X$*-probability* 1*. Therefore,* $\mathbb{P}^X$ *almost surely, the change of times* $F(x,t)$ *goes to* ∞ *as* $t \to \infty$ *for* μ *almost every* x*. Then use the coupling trick to deduce that the property holds for all starting points.*

3.3 Liouville Brownian motion on a bounded planar domain $D \subsetneqq \mathbb{C}$

Let $D \subsetneqq \mathbb{C}$ be a bounded simply connected domain of the plane. Let X be the GFF on D with Dirichlet boundary conditions and let $M = M_\gamma(X)$ denote the Liouville measure for $\gamma \in [0,2)$ on the domain D. We wish to briefly discuss how to construct a Liouville Brownian motion for $\gamma \in [0,2)$ on the domain D. We highlight two approaches.

3.3.1 Comparison with the massive Liouville Brownian motion on $\mathbb{R}^2$ through Kahane's convexity inequalities

In order to extend the approach we developed in $\mathbb{R}^2$ for the (massive)-free field to the case of our bounded domain D, one has to deal with two differences. First the field X in D with Dirichlet boundary conditions is no longer stationary in $x \in D$. Second, the quadratic variation $\langle \mathcal{B} \rangle_{\mathbf{t}}$ will no longer tend to infinity due to the fact that the LBM will eventually leave the domain D. To compare the situation in D with the situation in $\mathbb{R}^2$, endowed with a massive free field X_m, it is fruitful to rely on Kahane's convexity inequalities given in Lemma A.5. But to rely on these inequalities, one needs to show that for any $x, y \in D$, $\mathrm{Cov}\big[X(x), X(y)\big] \leqslant \mathrm{Cov}\big[X_m(x), X_m(y)\big]$. Recall that in a domain D (see for example [69]), one has

$$\mathrm{Cov}\big[X(x), X(y)\big] = G_D(x,y) = \log \frac{1}{|x-y|} + H^x(y)\,,$$

where G_D is the Green function of D and where $y \mapsto H^x(y)$ is the harmonic extension of the function $y \mapsto \log|x-y|$ on ∂D. In particular, one always has

$$\mathrm{Cov}\big[X(x), X(y)\big] \leqslant \frac{1}{|x-y|} + \log \operatorname{diam}(D)\,.$$

Now recall from subsection 2 that

$$\mathrm{Cov}\big[X_m(x), X_m(y)\big] = \log_+ \frac{1}{|x-y|} + g_m(x,y)\,,$$

where g_m is a continuous bounded function $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$. As in the Proof 2. that $F(x,t) \to \infty$ in subsection 2.4, let Y be an independent standard Gaussian variable of variance $\mathbb{E}\big[Y^2\big] = \log_+ \operatorname{diam}(D) + \|g_m\|_\infty$ in such a way that one has

$$\mathrm{Cov}\big[X\big] \leqslant \mathrm{Cov}\big[X_m + Y\big]\,.$$

The contribution of the independent variable Y will factorize in each required computation leading to an additional constant $c = c(Y)$. As such, by Lemma A.5, one can control all expectations of the form

$$\mathbb{E}\big[G(\int_D e^{\gamma X_r - \frac{\gamma^2}{2}\mathbb{E}\left[X_r^2\right]}\sigma(dr))\big]\,,$$

for any convex function $G : \mathbb{R}_+ \to \mathbb{R}$ with polynomial growth and for any $\gamma < 2$ (since we have estimates on such functionals for X_m only when $\gamma < 2$). By applying this Kahane convexity inequality to the occupation time measure σ of a Brownian motion $(B_t)_t$ stopped on ∂D, all the steps required to prove the analog of Theorem 1.2 can now be made rigorous. The proofs which analyze the behaviour of the quadratic variation of $\mathcal{B}_{\mathbf{t}}$ would require more effort in this case (in particular if $\gamma^2 \in [2,4)$ due to the negative moments in Lemma 2.13). Yet on the intuitive level, since the field X has Dirichlet boundary conditions, the Liouville Brownian motion should not be slowed down near ∂D (in general, the LBM is slowed down in areas where X is large). This means that one should have $\langle \mathcal{B}_{\mathbf{t}\wedge\tau_D}\rangle_\infty < \infty$ a.s.

3.3.2 Using a coupling argument

Let us briefly sketch a second reason why it is not hard to extend the above Liouville Brownian motions to the case of a bounded domain D.

Even though we did not find a proper reference, the following Lemma in the spirit of [27, section 4.5] and [68, section 3.1] should hold.

Lemma 3.2. *Let $D' \subset D$ be two domains with $\bar{D}' \subset D$. Let $T > 0$ such that $D \subset [-T,T]^2$. Furthermore, let X_D be the GFF in D with Dirichlet boundary conditions and let X_T denote the GFF with vanishing mean in the 2d-Torus with fundamental domain $[-T,T]^2$. Then the law of $(X_D)_{|D'}$ restricted to the domain D' is absolutely continuous with respect to the law of $(X_T)_{|D'}$.*

This lemma easily enables us to extend the above construction of a LBM on the $2T$-Torus to a LBM defined on D. Indeed, for each $\varepsilon > 0$, one applies the above lemma with $D' := D^{(\varepsilon)} := \{x \in D, \mathrm{dist}(x, \partial D) > \varepsilon\}$.

Let us end this section with the following remark.

Remark 3.3. *As is well known from Levy's theorem, standard two-dimensional Brownian motion is a conformally invariant object (modulo an explicit change of time-parametrization). Since the Liouville measures are conformally co-variant (see [32]), (which follows from the conformal invariance property of the GFF, see [69]), it is not hard to obtain that the Liouville Brownian motion is a conformally invariant object as well, up to a different time-parametrization which may be written explicitly.*

4 Liouville semi-group, invariant measure and associated Dirichlet form

This section is divided as follows. We start by defining the Liouville semi-group $(P^X_{\mathbf{t}})_{\mathbf{t} \geqslant 0}$ which is induced by the Liouville Brownian motion. We then show along subsection 4.2 that this semi-group is reversible with respect to the Liouville measure $M = M_\gamma$. Then, using classical arguments we show in subsection 4.3 that the semi-group P^x can be extended to a strongly-continuous semi-group on $L^p(\mathbb{R}^2, M)$ for all $1 \leqslant p < \infty$. Finally, we define in subsection 4.4 the Liouville Laplacian.

4.1 Definition of the Liouville semi-group

Definition 4.1. *The LBM is a time-homogeneous strong Markov Feller diffusion on $\mathbb{R}^2$. Therefore one can associate a Feller semi-group $(P^X_{\mathbf{t}})_{\mathbf{t}}$ acting on $C_b(\mathbb{R}^2)$.*

The Feller semi-group thus obtained is the limit of the n-regularized semigroup in the following sense.

Theorem 4.2. *Almost surely in X, the n-regularized semigroup $(P^n)_n$ converges point wise towards the Liouville semigroup. More precisely, for all bounded continuous function f, we have:*

$$\forall x \in \mathbb{R}^2, \forall t \geqslant 0, \quad \lim_{n \to \infty} P^n_{\mathbf{t}} f(x) = P^X_{\mathbf{t}} f(x).$$

Furthermore, for all bounded continuous function f, the mapping $(\mathbf{t}, x) \mapsto P^X_{\mathbf{t}} f(x)$ is continuous.

Proof. Since $P^n_{\mathbf{t}} f(x) = \mathbb{E}^x[f(\mathcal{B}^{n,x}_{\mathbf{t}})]$, the first statement is a direct consequence of Theorem 2.27. The second statement also results from Theorem 2.27 since $\mathcal{B}^x_{\mathbf{t}}$ converges in law towards $\mathcal{B}^{x_0}_{\mathbf{t}_0}$ when $(\mathbf{t}, x) \to (\mathbf{t}_0, x_0)$. $\square$

4.2 Reversibility of the Liouville measure under $(P^X_{\mathbf{t}})_{\mathbf{t}\geqslant 0}$

We wish to prove the following theorem.

Proposition 4.3. *If $\gamma < 2$, almost surely in X, the Liouville measure $M = M_\gamma$ is reversible under $P_{\mathbf{t}} = P^X_{\mathbf{t}}$. More exactly, for all functions f, g in $C_c(\mathbb{R}^2)$ and all $\mathbf{t} \geqslant 0$, we have*

$$\int_{\mathbb{R}^2} P^X_{\mathbf{t}} f(x) g(x) M(dx) = \int_{\mathbb{R}^2} f(x) P^X_{\mathbf{t}} g(x) M(dx).$$

Proof. For all $n \geqslant 1$ and every functions $f, g \in C_c(\mathbb{R}^2)$, we have:

$$\int_{\mathbb{R}^2} \mathbb{E}^B[f(\mathcal{B}^{n,x}_{\mathbf{t}})] g(x) M_n(dx) = \int_{\mathbb{R}^2} \mathbb{E}^B[g(\mathcal{B}^{n,x}_{\mathbf{t}})] f(x) M_n(dx). \tag{4.1}$$

Let us prove that we can pass to the limit as $n \to \infty$ in each side of the above relation. Recall that, almost surely in X, the measures $(M_n)_n$ weakly converge towards a Radon measure M.

$$\begin{aligned}
\int_{\mathbb{R}^2} P^X_{\mathbf{t}} f(x) g(x)\, M(dx) = & \int_{\mathbb{R}^2} P^X_{\mathbf{t}} f(x) g(x)\, M(dx) - \int_{\mathbb{R}^2} P^n_{\mathbf{t}} f(x) g(x)\, M(dx) \\
& + \int_{\mathbb{R}^2} P^n_{\mathbf{t}} f(x) g(x)\, M(dx) \\
\overset{def}{=} & E_n + \int_{\mathbb{R}^2} P^n_{\mathbf{t}} f(x) g(x)\, M(dx),
\end{aligned}$$

where we have set

$$E_n = \int_{\mathbb{R}^2} P^X_{\mathbf{t}} f(x) g(x)\, M(dx) - \int_{\mathbb{R}^2} P^n_{\mathbf{t}} f(x) g(x)\, M(dx).$$

By taking the conditional expectation, we get:

$$\begin{aligned}
& \mathbb{E}^X\Big[\int_{\mathbb{R}^2} P^X_{\mathbf{t}} f(x) g(x)\, M(dx) | \mathcal{F}_n\Big] \\
& \quad = \mathbb{E}^X\Big[\int_{\mathbb{R}^2} P^X_{\mathbf{t}} f(x) g(x)\, M(dx) - \int_{\mathbb{R}^2} P^n_{\mathbf{t}} f(x) g(x)\, M(dx) | \mathcal{F}_n\Big] \\
& \quad\quad + \mathbb{E}^X\Big[\int_{\mathbb{R}^2} P^n_{\mathbf{t}} f(x) g(x)\, M(dx) | \mathcal{F}_n\Big] \\
& \quad \overset{def}{=} \mathbb{E}^X\Big[E_n | \mathcal{F}_n\Big] + \int_{\mathbb{R}^2} P^n_{\mathbf{t}} f(x) g(x)\, M_n(dx).
\end{aligned}$$

Let us prove that the quantity $\mathbb{E}^X\Big[E_n|\mathcal{F}_n\Big]$ converges almost surely towards 0 up to extracting a subsequence. We have:

$$\mathbb{E}^X\Big[\Big|\mathbb{E}^X\Big[E_n|\mathcal{F}_n\Big]\Big|\Big] \leqslant \mathbb{E}^X\Big[|E_n|\Big].$$

From Theorem 2.27, observe that E_n almost surely converges to 0. Furthermore, the sequence $(E_n)_n$ is uniformly integrable since:

$$|E_n| \leqslant 2\|g\|_\infty \|f\|_\infty M(B(0,R))$$

for $R > 0$ large enough so as to make sure that $\mathrm{Supp}(f) \cup \mathrm{Supp}(g) \subset B(0,R)$. Uniform integrability then results from the relation $\mathbb{E}[M(B(0,R))^p] < +\infty$ for $0 < p < 4/\gamma^2$ (see [44]). We deduce that the sequence $(\mathbb{E}^X\big[E_n|\mathcal{F}_n\big])_n$ converges towards 0 in L^1. Furthermore, the sequence

$$\Big(\mathbb{E}^X\Big[\int_{\mathbb{R}^2} P^X_{\mathbf{t}} f(x)g(x)\, M(dx)|\mathcal{F}_n\Big]\Big)_n$$

is a uniformly integrable martingale, which converges almost surely and in L^1 towards $\int_{\mathbb{R}^2} P^X_{\mathbf{t}} f(x)g(x)\, M(dx)$. We deduce that the sequence with generic term

$$\int_{\mathbb{R}^2} P^X_{\mathbf{t}} f(x)g(x)\, M(dx) - \int_{\mathbb{R}^2} P^n_{\mathbf{t}} f(x)g(x)\, M_n(dx)$$

converges towards 0 in L^1. The same argument establishes that the sequence with generic term

$$\int_{\mathbb{R}^2} f(x)P^X_{\mathbf{t}} g(x)\, M(dx) - \int_{\mathbb{R}^2} f(x)P^n_{\mathbf{t}} g(x)\, M_n(dx)$$

converges towards 0 in L^1. From (4.1), we deduce that, almost surely in X,

$$\int_{\mathbb{R}^2} P^X_{\mathbf{t}} f(x)g(x)M(dx) = \int_{\mathbb{R}^2} f(x)P^X_{\mathbf{t}} g(x)M(dx). \tag{4.2}$$

Now we may assume that, almost surely in X, (4.2) holds for every rational $\mathbf{t} \geqslant 0$ and every functions f, g belonging to a countable family $(f_k)_k$ in $C_c(\mathbb{R}^2)$ dense in $C_c(\mathbb{R}^2)$ for the uniform topology. By continuity in $\mathbf{t}$ of the mapping $t \mapsto P_{\mathbf{t}} f(x)$, we deduce that, almost surely in X, (4.2) holds for all $\mathbf{t} \geqslant 0$. By density of the family $(f_k)_k$ in $C_c(\mathbb{R}^2)$, it is plain to deduce that (4.2) holds for all functions f, g in $C_c(\mathbb{R}^2)$. □

4.3 Extension of the semi-group to $L^p(\mathbb{R}^2, M)$ and its strong-continuity

From the reversibility of the Liouville measure M, one obtains the following result.

Corollary 4.4. *For $\gamma < 2$, the Liouville semi-group $(P^X_{\mathbf{t}})_{\mathbf{t} \geqslant 0}$ extends to a semi-group on $L^p(\mathbb{R}^2, M)$ for all $1 \leqslant p < +\infty$. Furthermore, the semi-group is strongly continuous for $1 < p < +\infty$.*

Proof. From Proposition 4.6, we get for all $\mathbf{t} \geqslant 0$ and $f, g \in C_c(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2} P^X_{\mathbf{t}} f(x) g(x) M(dx) = \int_{\mathbb{R}^2} f(x) P^X_{\mathbf{t}} g(x) M(dx).$$

If we assume that f, g are nonnegative and $1 \leqslant p < +\infty$, we deduce from the Jensen inequality:

$$\begin{aligned}\int_{\mathbb{R}^2} (P^X_{\mathbf{t}} f(x))^p g(x) M(dx) &\leqslant \int_{\mathbb{R}^2} P^X_{\mathbf{t}} f^p(x) g(x) M(dx) \\ &= \int_{\mathbb{R}^2} f^p(x) P^X_{\mathbf{t}} g(x) M(dx) \\ &\leqslant \|g\|_\infty \int_{\mathbb{R}^2} f(x)^p M(dx). \end{aligned} \tag{4.3}$$

Since the measure M is Radon over $\mathbb{R}^2$, which is a σ-compact complete space, M is a regular measure. We deduce in a standard way that:

Lemma 4.5. *Almost surely in X, for all $1 \leqslant p < +\infty$, we have $C_c(\mathbb{R}^2) \subset L^p(\mathbb{R}^2, M)$ and $C_c(\mathbb{R}^2)$ is dense in $L^p(\mathbb{R}^2, M)$.*

Now, if we consider a continuous function $0 \leqslant g \leqslant 1$ worth 1 over $B(0, R)$ in (4.3), and if we let R go to ∞, we deduce

$$\int_{\mathbb{R}^2} (P^X_{\mathbf{t}} f(x))^p M(dx) \leqslant \int_{\mathbb{R}^2} f(x)^p M(dx). \tag{4.4}$$

Lemma 4.5 together with (4.4) entails that (4.4) remains true for all $f \in L^p(\mathbb{R}^2, M)$. Therefore, for all $\mathbf{t} \geqslant 0$, P^X_t continuously maps $L^p(\mathbb{R}^2, M)$ into $L^p(\mathbb{R}^2, M)$. Furthermore, it is a contraction (not necessarily strict).

It remains to prove that the semi-group P^X is strongly continuous for $1 \leqslant p < +\infty$. From Lemma 4.5 again and because $(P^X_{\mathbf{t}})_{\mathbf{t}}$ is a contraction semigroup for $1 \leqslant p < +\infty$, it suffices to prove that $P^X_{\mathbf{t}} f$ strongly converges towards f in $L^p(\mathbb{R}^2, M)$ as $\mathbf{t} \to 0$ for all continuous function f with compact support.

Let us first treat the case $p = 2$. We have:

$$\begin{aligned}&\int_{\mathbb{R}^2} |P^X_{\mathbf{t}} g(x) - g(x)|^2 \, M(dx) \\ &= \int_{\mathbb{R}^2} |P^X_{\mathbf{t}} g(x)|^2 + |g(x)|^2 \, M(dx) - 2 \int_{\mathbb{R}^2} P^X_{\mathbf{t}} g(x) \, g(x) \, M(dx) \\ &\leqslant 2 \int_{\mathbb{R}^2} |g(x)|^2 \, M(dx) - 2 \int_{\mathbb{R}^2} P^X_{\mathbf{t}} g(x) \, g(x) \, M(dx). \end{aligned} \tag{4.5}$$

Because g is continuous, Theorem 2.27 ensures that $P^X_{\mathbf{t}} g(x)$ converges towards $g(x)$ as $\mathbf{t} \to 0$. The dominated convergence theorem then implies that

$$\int_{\mathbb{R}^2} P^X_{\mathbf{t}} g(x) \, g(x) \, M(dx) \to \int_{\mathbb{R}^2} |g(x)|^2 \, M(dx), \quad t \to 0. \tag{4.6}$$

By gathering (4.5)+(4.6), we get the strong continuity for $p=2$.

To obtain the strong continuity for $p>2$, we just observe that:

$$\begin{aligned}\int_{\mathbb{R}^2}|P^X_{\mathbf{t}}g(x)-g(x)|^p\,M(dx)&\leqslant\|P^X_tg-g\|_\infty^{p-2}\int_{\mathbb{R}^2}|P^X_{\mathbf{t}}g(x)-g(x)|^2\,M(dx)\\&\leqslant 2\|g\|_\infty^{p-2}\int_{\mathbb{R}^2}|P^X_{\mathbf{t}}g(x)-g(x)|^2\,M(dx).\end{aligned}$$

For $1<p<2$, we use the relation, valid for $\epsilon>0$,

$$\begin{aligned}&\int_{\mathbb{R}^2}|P^X_{\mathbf{t}}g(x)-g(x)|^p\,M(dx)\\&=\int_{\{|P^X_{\mathbf{t}}g(x)-g(x)|\,\geqslant\,\epsilon\}}|P^X_{\mathbf{t}}g(x)-g(x)|^p\,M(dx)+\int_{\{|P^X_{\mathbf{t}}g(x)-g(x)|<\epsilon\}}|P^X_{\mathbf{t}}g(x)-g(x)|^p\,M(dx)\\&\leqslant\epsilon^{p-2}\int_{\mathbb{R}^2}|P^X_{\mathbf{t}}g(x)-g(x)|^2\,M(dx)+\epsilon^{p-1}\int_{\mathbb{R}^2}|P^X_{\mathbf{t}}g(x)-g(x)|\,M(dx),\end{aligned}$$

from which we deduce:

$$\limsup_{t\to 0}\int_{\mathbb{R}^2}|P^X_{\mathbf{t}}g(x)-g(x)|^p\,M(dx)\leqslant 2\epsilon^{p-1}\int_{\mathbb{R}^2}|g(x)|\,M(dx).$$

Since ϵ can be chosen arbitrarily small.

For $p=1$, we choose a closed ball B centered at 0 and containing $\mathrm{Supp}(g)$. Then we choose a continuous function ϕ such that $0\leqslant\phi\leqslant 1$, $\phi(x)=1$ for $x\in B^c$ and $\phi(x)=0$ for $\mathrm{dist}(x,B^c)\leqslant 1$.

$$\begin{aligned}\int_{\mathbb{R}^2}|P^X_{\mathbf{t}}g(x)-g(x)|\,M(dx)\leqslant&\int_B|P^X_{\mathbf{t}}g(x)-g(x)|\,M(dx)\\&+\int_{B^c}|P^X_{\mathbf{t}}g(x)|\,M(dx).\end{aligned}$$

The first term in the right-hand side converges to 0 as $\mathbf{t}\to 0$. Concerning the second term, we have:

$$\begin{aligned}\int_{B^c}|P^X_{\mathbf{t}}g(x)|\,M(dx)&\leqslant\int_{\mathbb{R}^2}P^X_{\mathbf{t}}|g|(x)\phi(x)\,M(dx)\\&\leqslant\int_{\mathbb{R}^2}|g|(x)P^X_{\mathbf{t}}\phi(x)\,M(dx).\end{aligned}$$

The dominated convergence theorem entails that the latter quantity converges towards $\int_{\mathbb{R}^2}|g|(x)\phi(x)\,M(dx)$ as $t\to 0$, which is less than $\int_{\{\mathrm{dist}(x,B^c)\,\leqslant\,1\}}|g|(x)\,M(dx)$. By choosing B arbitrarily big, we complete the proof. □

As a straightforward consequence of Proposition 4.3 and Corollary 4.4, we get:

Theorem 4.6. *For $\gamma<2$, almost surely in X, the massive Liouville Brownian motion is reversible w.r.t. the Liouville measure. Equivalently, the Liouville semigroup on $L^2(\mathbb{R}^2,M)$ is self-adjoint. The Liouville measure is therefore invariant for the Liouville Brownian motion.*

So our Liouville Brownian motion a.s. preserves the Liouville measure. It would be interesting to check that the semigroup converges towards the Liouville measure and to investigate at which rate this convergence occurs.

4.4 Liouville Laplacian

For $\gamma < 2$, the **Liouville Laplacian** Δ_X is defined as the generator of the Liouville Brownian motion times the usual extra factor $\sqrt{2}$. The Liouville Laplacian corresponds to an operator which can formally be written as

$$\Delta_X = e^{-\gamma X(x) + \frac{\gamma^2}{2}\mathbb{E}[X(x)^2]}\Delta$$

and can be thought of as the Laplace-Beltrami operator of $2d$-Liouville quantum gravity.

Finally we conclude this section with a remark about the **fractional Liouville Laplacian**. Indeed, one may also wishes to define rigorously the fractional Liouville Laplacian for $0 < \alpha < 1$. The underlying Markov process can be obtained by subordinating the Liouville Brownian motion with an independent α-stable Lévy subordinator. The fractional Liouville Laplacian is then nothing but the generator of this Markov process.

Let us try to give an intuition of this operator. The Euclidean fractional Laplacian in dimension 2 can be formally written as

$$(-\triangle)^\alpha f(x) = P.V.\ c_\alpha \int_{\mathbb{R}^2} \frac{f(x+z) - f(x)}{|z|^{2+2\alpha}} dz.$$

On a Riemannian manifold the above expression remains true provided that we replace the quantity "$x+z$" by the exponential map of the manifold (see for instance [6] and references therein). In Liouville quantum gravity, this expression should remain valid, provided that one can give sense to the exponential map.

4.5 Dirichlet form

One may also attach to the Liouville semigroup $(P^X_{\mathbf{t}})_{\mathbf{t}\geqslant 0}$ a Dirichlet form by the formula:

$$\Sigma(f,f) = \lim_{\mathbf{t}\to 0} \frac{1}{\mathbf{t}} \int \big(f(x) - P^X_{\mathbf{t}} f(x)\big) f(x) M(dx). \tag{4.7}$$

with domain $\mathcal{F}$, which is defined as the set of functions $f \in L^2(\mathbb{R}^2, M)$ for which the above limit exists and is finite. A thorough study of this Dirichlet form is carried out in [39].

5 Remaining conjectures and open problems

5.1 About the construction for all possible values of γ^2

Question 1. Warm-up in the supercritical regime. Prove that for $\gamma^2 \geqslant 4$, the changes of times $(F^n)_n$ converge to 0. This intuitively means that, for $\gamma^2 \geqslant 4$, the Liouville Brownian motion is not defined because it cannot be stabilized: its speed instantaneously blows up.

Question 2. Derivative Liouville PCAF. For $\gamma^2 = 4$, construct the critical Liouville Brownian motion in the spirit of [30, 31] (which contain many other references in closely related contexts). In particular, construct the derivative change of times

$$F(x,t) = \int_0^t \big(2\mathbb{E}[X(x+B_r)^2] - X(x+B_r)\big) e^{2X(x+B_r) - 2\mathbb{E}[X(x+B_r)^2]}\, dr.$$

Question 3. About the maximum of the GFF. Determine the asymptotic behaviour of the maximum of the GFF along the Brownian curve. More precisely, in section 2.1, choose the sequence $(c_n)_n$ in order to make sure that $\mathrm{Var}(X_n) = \ln n$. Is it true that, for $t > 0$ the family

$$\max_{0 \leqslant s \leqslant t} X_n(x+B_s) - 2\ln n + c \ln\ln n$$

converges in law as $n \to \infty$ for some $c \in \mathbb{R}$? Can one express the limiting law as a shift of a Gumbel law by the (log) derivative time change of Question 2? Formulate a similar conjecture/result for the maximum of a discrete GFF along the path of a simple random walk on the vertices of a regular lattice of $\mathbb{R}^2$.

Question 4. Duality. For $\gamma^2 > 4$, construct the dual Liouville Brownian motion in the spirit of [9].

5.2 Structure of the Liouville Brownian motion

Question 5. Heat kernel. Prove that the Liouville heat kernel $p^X(\mathbf{t}, x, y)$ is continuous with respect to x, y and that it is positive. It is not clear that we can get further "Euclidean" regularity. C^∞-smoothness is more likely to make sense for the Liouville metric. Furthermore, characterize the Liouville heat kernel $p^X(\mathbf{t}, x, y)$ as the minimal fundamental solution of the Liouville heat equation. Also, prove rigorously that the spectral dimension of two-dimensional quantum gravity is 2 (see [4]) with the help of the heat kernel. We further stress that several important properties of $2d$ Liouville quantum gravity are heuristically derived from the asymptotic behaviour of the heat kernel for small times in [4, 5, 17, 18, 22, 76].

Question 6. About the role of the dimension. In dimension 2, the "dimension" of the Brownian curve is the same as that of the space, namely 2 (of course,

this is also valid in dimension 1 but we do not discuss the one-dimensional case here). Therefore Kahane's theory applies for the construction of the Liouville Brownian motion for the same values of the parameter γ as those of the construction of the Liouville measure, i.e. for $\gamma \in [0,2[$. This is the miracle of dimension 2. In higher dimensions, the situation becomes more interesting. Indeed, the construction of the measure is valid for $\gamma^2 < 2d$ whereas the dimension of the Brownian curve still remains equal to 2, yielding a construction for $\gamma^2 < 4$. So there is a gap, for $4 \leqslant \gamma^2 < 2d$, where a construction (even at a heuristic level) of the Liouville Brownian motion is not obvious. Does it produce new qualitative behaviors? The study of the Liouville measures has shown that a very rich panel of different behaviors occurs in the super-critical case $\gamma^2 \geqslant 2d$ (see [9, 30, 31] based on similar discrete models [35]). It is therefore natural to wonder whether the situation $4 \leqslant \gamma^2 < 2d$ for the Liouville Brownian motion yields a qualitative behavior similar to that of super-critical measures or whether this situation produces quite a new object.

Question 7. Liouville Brownian bridges. Let us consider a Liouville Brownian bridge from x to y with lifetime t, call it $\mathcal{B}^{x,y,\mathbf{t}}$. It can be obtained by conditioning the Liouville Brownian motion starting from x to reach y at time $\mathbf{t}$. Prove that the family of curves $\{\mathcal{B}^{x,y,\mathbf{t}}_{st}; 0 \leqslant s \leqslant 1\}$ is sequentially compact as $\mathbf{t} \to 0$ and that any possible limit is a minimizing geodesic between x and y.

5.3 Fractal geometry of Liouville quantum gravity

Question 8. Brownian formulation of the KPZ formula. We want here to address the issue of formulating a Brownian version of the KPZ formula. Remind that the KPZ formula is a relation between the fractal dimensions of sets as seen by the Euclidean metric or the quantum geometry. In [32, 65], the KPZ formula is proved by treating the field $e^{\gamma X}$ as a random measure, but not as a random metric. Another derivation of the KPZ formula is suggested in [22]: express the KPZ formula in terms of time spend by the Liouville Brownian motion to cover Euclidean balls. We want to describe below a possible way of making this rigorous, which differs from the approach developed in [22].

The quantum mass that we assign to a ball of Euclidean radius R will be the time spent by the Liouville Brownian motion to leave this ball. If one wants to figure out what this quantum time looks like, then one has to introduce the stopping time

$$\tau^R_{\mathrm{LQG}}(x) = \inf\{u \geqslant 0; \mathcal{B}^x_u \notin B(x,R)\}.$$

This quantity stands for the time spent by the LBM to leave the Euclidean ball of radius R. It can be expressed in terms of the time spent by the Euclidean Brownian motion to leave this ball

$$\tau^R_{\mathrm{E}}(x) = \inf\{u \geqslant 0; x + B_u \notin B(x,R)\},$$

by the relation

$$F(x, \tau^R_{\mathrm{E}}) = \tau^R_{\mathrm{LQG}}(x).$$

One could thus assign to the Euclidean ball of radius R centered at x the quantum mass

$$\mathbb{E}^B[\tau^R_{\mathrm{LQG}}(x)].$$

We then naturally define the following quantum s-dimensional Hausdorff measure of a set K:

$$H^s_{LQG}(A)=\lim_{\delta\to 0}H^{s,\delta}_{LQG}(A),\quad \text{where } H^{s,\delta}_{LQG}(K)=\inf\big\{\sum_k \mathbb{E}^B[\tau^{r_k}_{\mathrm{LQG}}(x_k)]^s\big\}$$

where the infimum runs over all the covering $(B(x_k,r_k))_k$ of K with closed Euclidean balls with radius $r_k\leqslant\delta$. H^s_{LQG} is a metric outer measure on $\mathbb{R}^d$, for which the Borelian sets are H^s_{LQG}-measurable. The Quantum Hausdorff dimension $\dim_{LGQ}(K)\in[0,1]$ of the set K is then defined as the value

$$\dim_{LGQ}(A)=\inf\{s\geqslant 0;\ H^s_{LGQ}(K)=0\}=\sup\{s\geqslant 0;\ H^s_{LGQ}(K)=+\infty\}. \quad (5.1)$$

The question is then to prove that (if possible almost surely in X)

$$\dim_E(K)=\big(1+\frac{\gamma^2}{4}\big)\dim_{\mathrm{LQG}}(K)-\frac{\gamma^2}{4}\dim_{\mathrm{LQG}}(K)^2,$$

where $\dim_E(K)$ stands for the standard Euclidean Hausdorff dimension of K.

5.4 Connection with Brownian motion on planar maps

Question 9. Random walks on planar maps. Prove the convergence of Brownian motion on planar maps towards the Liouville Brownian motion as explained in conjecture 1.

5.5 Related stochastic calculus

Question 10. Quantum BSDEs Prove a martingale representation theorem for the Liouville Brownian motion. If true, this opens the doors of a theory of Backward Stochastic Differential Equations (BSDE) with respect to the Liouville Brownian motion. In particular, one should be able to give a probabilistic representation of nonlinear problems of the type:

$$\partial_t u=\triangle_X u+f(x,u,\nabla_X u)$$

with suitable initial condition $u(0,\cdot)$.

5.6 Related models

Question 11. Liouville Brownian motion and cascades. Construct the analog of the Liouville Brownian motion in the context of Mandelbrot's multiplicative

cascades (see [8, 45, 46] or [10, 13] in the KPZ context) or even Branching Random Walks. Beyond the fact that it is a toy model, it is a powerful laboratory to understand the continuous case. Though independence structure is reinforced in this model, it has been for long illustrated that the main qualitative phenomena observed in the context of cascades remain true in the continuous case. Furthermore, Kahane's convexity inequalities provide a (one-sided) powerful bridge between cascades and multiplicative chaos (see [44, 30] for instance).

Question 12. Log-infinitely divisible geometries. Construct the analog of the Liouville Brownian motion with respect to log-correlated infinitely divisible fields instead of the GFF or MFF. The reader is referred to [7, 11, 36, 63, 64] for insights of the topic. We also would like to ask the following question: what is the planar maps counterpart of these (non-lognormal) other geometries?

A Background about Gaussian multiplicative chaos theory

Here we recall a few material about Gaussian multiplicative chaos theory that can be found in [44]. First we remind of the following regularity notion for a measure

Definition A.1. *A finite measure σ on (a bounded domain of) $\mathbb{R}^d$ is said to be in the class R_α for $\alpha > 0$ if for all $\epsilon > 0$ there is $\delta > 0$ and a compact set $A \subset D$ such that $\sigma(D \setminus A) \leqslant \epsilon$ and:*

$$\forall O \text{ open set}, \quad \sigma(O \cap A) \leqslant C \mathrm{diam}(O)^{\alpha+\delta}, \tag{A.1}$$

where $\mathrm{diam}(O)$ *stands for the Euclidean diameter of O.*

Remark A.2. *To be precise, Kahane considered the class R_α of measures supported by a compact set. Actually his proofs straightforwardly adapt to the case when the measures are not necessarily compactly supported, but finite.*

Now we consider a Radon measure σ on (a bounded domain of) $\mathbb{R}^d$ and the associated chaos:

$$M_\sigma(dx) = \lim_{n\to\infty} M_\sigma^n(dx),$$

where

$$M_\sigma^n(dx) = \int_{\cdot} e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}\,\sigma(dx).$$

Recall (see [44]):

Theorem A.3 (Law of the chaos)**.** *The law of the measure M_σ does not depend on the choice of the decomposition of the covariance kernel of the Gaussian field X into a sum of nonnegative continuous kernels of positive type.*

We stress that the only point where Kahane's theory uses positivity of the kernels in to prove this uniqueness property. All the remaining machinery works without assuming this condition. This is worth being mentioned to apply Kahane's theory to the GFF on the sphere or the torus: the eigenvalues take possibly negative values. Nevertheless, further reinforcements of the above theorem are established in [66]. In particular, we can deal with kernels that may take negative values and prove uniqueness in law whatever the chosen decomposition of the Free Field, and more generally the underlying log-correlated Gaussian field.

Theorem A.4 (Non-degeneracy). *Assume that the measure σ is in the class R_α for some $\alpha > 0$. If $\gamma^2 < 2\alpha$ then, for all Borelian set A with finite σ-measure, the sequence $(M_\sigma^n(A))_n$ is uniformly integrable. Furthermore, the chaos M is non degenerate and belongs to $R_{\alpha-\frac{\gamma^2}{2}}$.*

Lemma A.5. *Let $F:\mathbb{R}_+\to\mathbb{R}$ be some convex function such that*

$$\forall x\in\mathbb{R}_+,\quad |F(x)|\leqslant M(1+|x|^\beta),$$

for some positive constants M,β, and σ be a Radon measure on the Borelian subsets of a locally compact separable metric space (D,d). Given two continuous centered Gaussian processes $(X_r)_{r\in D},(Y_r)_{r\in D}$ with continuous covariance kernels k_X and k_Y such that

$$\forall u,v\in D,\quad k_X(u,v)\leqslant k_Y(u,v).$$

Then

$$\mathbb{E}\Big[F\Big(\int_D e^{X_r-\frac12\mathbb{E}[X_r^2]}\,\sigma(dr)\Big)\Big]\leqslant\mathbb{E}\Big[F\Big(\int_D e^{Y_r-\frac12\mathbb{E}[Y_r^2]}\,\sigma(dr)\Big)\Big].$$

Star scale invariance

The need of characterizing Gaussian multiplicative chaos may be achieved via a functional equation, called lognormal $\star$-scale invariance [2]:

Definition A.6. Log-normal $\star$-scale invariance. *A random Radon measure M is lognormal $\star$-scale invariant if for all $0<\varepsilon\leqslant 1$, M obeys the cascading rule*

$$\big(M(A)\big)_{A\in\mathcal{B}(\mathbb{R}^d)}\overset{law}{=}\Big(\int_A e^{\omega_\varepsilon(x)}M^\varepsilon(dx)\Big)_{A\in\mathcal{B}(\mathbb{R}^d)}\tag{A.2}$$

where ω_ε is a stationary stochastically continuous Gaussian process and M^ε is a random measure independent from ω_ε satisfying the scaling relation

$$\big(M^\varepsilon(A)\big)_{A\in\mathcal{B}(\mathbb{R}^d)}\overset{law}{=}\big(M(\frac{A}{\varepsilon})\big)_{A\in\mathcal{B}(\mathbb{R}^d)}.\tag{A.3}$$

□

Notice that the process ω_ε is unknown. Roughly speaking, we look for random measures that scale with an independent lognormal factor on the whole space. This property is shared by many examples of Gaussian multiplicative chaos as we will see below, but not all. And for those Gaussian multiplicative chaos that do not share this property, they are very close to satisfying it. If the reader is familiar with branching random walks (BRW), here is an explanation that may help intuition. If we consider a BRW the reproduction law of which does not change with time (i.e. is the same at each generation), the law of the branching random walk will be characterized by a discrete version of the above $\star$-scale invariance (in the lognormal case of course). If the reproduction law evolves in time, then we have to change things a bit to adapt to this time evolution. The same argument holds for the log-normal $\star$-scale invariance: it characterizes these Gaussian multiplicative chaos that do not vary along scales.

It is proved in [2] that a Gaussian multiplicative chaos

$$M(A) = \int_A e^{X_x - \frac{1}{2}\mathbb{E}[X_x^2]}\,dx$$

with respect to a stationary Gaussian field X with covariance kernel of the type

$$K(x,y) = \int_1^{+\infty} \frac{k((x-y)u)}{u}\,du, \tag{A.4}$$

for some continuous covariance function k such that $k(0) \leqslant \frac{2d}{1+\delta}$ (for some $\delta > 0$), is star scale invariant and non trivial. The converse is also studied in [2] and it is proved that lognormal star scale invariant measures are essentially of this form provided that the measure M possesses a moment of order $1+\delta$ for some $\delta > 0$. When a kernel K takes on the form A.4, it will be said star scale invariant. Observe that such kernels satisfy the scaling relation for all $\epsilon \in]0,1]$:

$$K(x,y) = K(\frac{x}{\epsilon},\frac{y}{\epsilon}) + k_\epsilon(x,y), \quad \text{with } k_\epsilon(x,y) = \int_1^{\frac{1}{\epsilon}} \frac{k((x-y)u)}{u}\,du.$$

In particular, we have:

$$K(x,y) \leqslant K(\frac{x}{\epsilon},\frac{y}{\epsilon}) + k(0)\ln\frac{1}{\epsilon}. \tag{A.5}$$

This relation turns out to be very useful when combined with Kahane's convexity inequalities (Lemma A.5).

B Finiteness of the moments

In this section, our only goal is to prove that

$$\mathbb{E}^X\mathbb{E}^B[F(x,t)^p] < +\infty$$

for $p \in [0, 4/\gamma^2[\cap[0,2]$. Of course, it suffices to compute the moments of order 2 when $\gamma^2 < 2$: this situation is trivial. So, due to their technicality, the following computations only make sense for $2 \leqslant \gamma^2 < 4$.

When p is less than 1, finiteness of the moments directly result from the uniform integrability of the sequence $(F^n(x,t))_n$. So it remains to investigate the case when $1 < p < 4/\gamma^2$ (and therefore $p < 2$). As $(F^n(x,t))_n$ is a uniformly integrable martingale, it suffices to prove $\mathbb{E}^X\mathbb{E}^B[F(x,t)^p] < +\infty$. Furthermore, by stationarity of the field X, we may assume that $x = 0$. By using the concavity of the mapping $x \mapsto x^{p/2}$ and the Jensen inequality, we get:

$$\begin{aligned}
\mathbb{E}^X\mathbb{E}^B\Big[\Big(F(0,t)\Big)^p\Big] &\leqslant \mathbb{E}^X\Big[\mathbb{E}^B\Big[F(0,t)^2\Big]^{p/2}\Big]\\
&= \mathbb{E}^X\Big[\Big(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_0^t\int_0^t e^{-\frac{|x|^2}{2s}-\frac{|y-x|^2}{2|r-s|}}\,\frac{drds}{4\pi^2 s|r-s|}M(dx)M(dy)\Big)^{p/2}\Big]\\
&\leqslant \mathbb{E}^X\Big[\Big(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x,y)M(dx)M(dy)\Big)^{p/2}\Big],
\end{aligned}$$

where we have set

$$f(x,y) = \int_0^t\int_s^t e^{-\frac{|x|^2}{2s}-\frac{|y-x|^2}{2|r-s|}}\,\frac{drds}{4\pi^2 s|r-s|}.$$

So one just has to prove that the expectation in the above right-hand side is finite. This is what we are going to prove.

In what follows, ξ_M stands for the structure exponent of the measure M. Recall that, in dimension d, it reads

$$\xi_M(p) = (d+\frac{\gamma^2}{2})p - \frac{\gamma^2}{2}p^2.$$

Of course, we can take here $d = 2$. But is is worth recalling this fact since it will happen that some arguments below will be carried out in dimension 1. So the reader will take care of replacing d by 1 when reading a proof in dimension 1. The main idea of our proof is the following. First, we observe that the function f possesses singularities. They are logarithmic (see below) when x or $|x-y|$ is close to 0. So we have to provide estimates on M that ensure that this logarithmic divergence can be overcome by the measure M. So, we will also have to treat the behaviour near infinity. We will split the space $\mathbb{R}^2\times\mathbb{R}^2$ into 3 domains $\{|x|\leqslant 1, |x-y|\leqslant 1\}$, $\{|x|\geqslant 1, |x-y|\leqslant 1\}$ and $\{|x-y|\geqslant 1\}$.

Domain $\{|x|\leqslant 1, |x-y|\leqslant 1\}$

We claim:

Lemma B.1. *For any $\gamma^2 < 4$ and $p \in]1, \frac{4}{\gamma^2}[$, there exists $\delta > 0$ such that*

$$\mathbb{E}\Big[\Big(\int_{\max(|x|,|y|)\leqslant 1}\frac{1}{|x|^\delta|x-y|^\delta}M(dx)M(dy)\Big)^{p/2}\Big] < +\infty.$$

To prove this lemma, we first prove

Lemma B.2. *For any $\gamma^2 < 4$ and $p \in]1, \frac{4}{\gamma^2}[$, there exists $\delta > 0$ such that*

$$\mathbb{E}\Big[\Big(\int_{\max(|x|,|y|)\,\leqslant\,1} \frac{1}{|x|^\delta} M(dx)M(dy)\Big)^{p/2}\Big] < +\infty.$$

Furthermore, there exists a constant $C > 0$ such that for all n:

$$\mathbb{E}\Big[\Big(\int_{\max(|x|,|y|)\,\leqslant\,2^{-n}} \frac{1}{|x|^\delta} M(dx)M(dy)\Big)^{p/2}\Big] \leqslant C 2^{-n(\xi_M(p)-\frac{\delta p}{2})}.$$

Proof. We carry out the proof in dimension 1 since, apart from notational issues, the dimension 2 does not raise any further difficulty. In that case, we have to prove

$$\mathbb{E}\Big[\Big(\int_{(x,y)\in[0,1]^2} \frac{1}{|x|^\delta} M(dx)M(dy)\Big)^{p/2}\Big] < +\infty.$$

Furthermore, from Kahane's convexity inequalities, it suffices to prove the above lemma for any log-correlated Gaussian field. We choose the perfect kernel of [7] (the reader may consult [64] to adapt the proof to higher dimension). We will only use scaling properties of this kernel. We further stress that a direct argument may be carried out with the help of Kahane's convexity inequalities (direct means without using the perfect kernel). We also remind the reader that the above integral is finite for $\delta = 0$ (see [44]). Therefore, by using sub-additivity of the mapping $x \mapsto x^{p/2}$, we have:

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{(x,y)\in[0,1]^2} \frac{1}{|x|^\delta} M(dx)M(dy)\Big)^{p/2}\Big] \\
&\qquad\qquad = \mathbb{E}\Big[\Big(\sum_{n=0}^{\infty} \int_{2^{-n-1}\,\leqslant\, x\,\leqslant\, 2^{-n}} \frac{1}{|x|^\delta} M(dx)M(dy)\Big)^{p/2}\Big] \\
&\qquad\qquad \leqslant \sum_{n=0}^{\infty} \mathbb{E}\Big[\Big(\int_{2^{-n-1}\,\leqslant\, x\,\leqslant\, 2^{-n}} \frac{1}{|x|^\delta} M(dx)M(dy)\Big)^{p/2}\Big] \\
&\qquad\qquad \leqslant \sum_{n=0}^{\infty} 2^{\delta(n+1)} \mathbb{E}\Big[\Big(M([2^{-n-1},2^{-n}])M([0,1])\Big)^{p/2}\Big].
\end{aligned}$$

Now we use the standard inequality $ab \leqslant \epsilon a^2 + \frac{b^2}{\epsilon}$ for any $\epsilon > 0$ and sub-additivity of the mapping $x \mapsto x^{p/2}$ to get:

$$(ab)^{p/2} \leqslant \epsilon^{p/2} a^p + \epsilon^{-p/2} b^p.$$

Therefore, with $a = M([2^{-n-1}, 2^{-n}])$, $b = M([0,1])$ and $\epsilon = 2^{(n+1)\xi_M(p)/p}$, we obtain:

$$\begin{aligned}
\mathbb{E}\Big[\Big(M([2^{-n-1},2^{-n}])M([0,1])\Big)^{p/2}\Big] \leqslant\ & (2^{(n+1)\xi_M(p)/p})^{p/2}\mathbb{E}\Big[M([2^{-n-1},2^{-n}])^p\Big] \\
& + (2^{(n+1)\xi_M(p)/p})^{-p/2}\mathbb{E}\Big[M([0,1])^p\Big].
\end{aligned}$$

By using now the exact scale invariance of the measure M, we get

$$\mathbb{E}\Big[M([2^{-n-1},2^{-n}])^p\Big]=2^{-(n+1)\xi_M(p)}\mathbb{E}\Big[M([0,1])^p\Big]$$

and plugging this relation into the above expression yields:

$$\mathbb{E}\Big[\Big(M([2^{-n-1},2^{-n}])M([0,1])\Big)^{p/2}\Big]\leqslant 2\times 2^{-(n+1)\xi_M(p)/2}\mathbb{E}\Big[M([0,1])^p\Big].$$

To sum up, we have:

$$\mathbb{E}\Big[\Big(\int_{(x,y)\in[0,1]^2}\frac{1}{|x|^\delta}M(dx)M(dy)\Big)^{p/2}\Big]\leqslant\sum_{n=0}^{\infty}2^{-(n+1)(\xi_M(p)/2-\delta)}\times 2\mathbb{E}\Big[M([0,1])^p\Big].$$

So, δ can clearly be chosen small enough to make the above series convergent.

The second statement results from the finiteness of the expectation that we have just proved and a straightforward scaling argument. □

Lemma B.3. *For any $\gamma^2<4$ and $p\in]1,\frac{4}{\gamma^2}[$, there exist $\delta>0$ and a constant $C>0$ (only depending on $\mathbb{E}\big[M\big([0,1]\big)^p\big]$) such that for all n:*

$$\mathbb{E}\Big[\Big(\int_{\substack{\max(|x|,|y|)\leqslant 1\\ 2^{-n-1}\leqslant|x-y|\leqslant 2^{-n}}}\frac{1}{|x|^\delta}M(dx)M(dy)\Big)^{p/2}\Big]\leqslant\frac{C}{1-\delta}2^{-n(\xi_M(p)-2-\frac{\delta p}{2})}.$$

Proof. Once again, we carry out the proof in dimension 1 since, apart from notational issues, the dimension 2 does not raise any further difficulty. In that case, we have to prove

$$\mathbb{E}\Big[\Big(\int_{\substack{(x,y)\in[0,1]^2\\ 2^{-n-1}\leqslant|x-y|\leqslant 2^{-n}}}\frac{1}{|x|^\delta}M(dx)M(dy)\Big)^{p/2}\Big]\leqslant C2^{-n(\xi_M(p)-1)}.$$

Furthermore, from Kahane's convexity inequalities, it suffices to prove the above lemma for any log-correlated Gaussian field. Once again, we choose the perfect kernel of [7] and we will only use scaling properties of this kernel. A direct argument may be again carried out with the help of Kahane's convexity inequalities. We will use the following elementary geometric argument: for any $n\geqslant 1$, the set of points 2^{-n}-close to the diagonal

$$\{(x,y)\in[0,1]^2;|x-y|\leqslant 2^{-n}\}$$

is entirely recovered by the union for $k=0,\dots,2^n-2$ of the (overlapping) squares

$[\frac{k}{2^n}, \frac{k+2}{2^n}]^2$. Therefore, by using sub-additivity of the mapping $x \mapsto x^{p/2}$, we have:

$$\mathbb{E}\Big[\Big(\int_{\substack{x,y\in[0,1] \\ 2^{-n-1} \leqslant |x-y| \leqslant 2^{-n}}} \frac{1}{|x|^\delta} M(dx)M(dy)\Big)^{p/2}\Big]$$
$$\leqslant \mathbb{E}\Big[\Big(\sum_{k=0,\dots,2^n-2} \int_{x,y\in[\frac{k}{2^n},\frac{k+2}{2^n}]^2} \frac{1}{|x|^\delta} M(dx)M(dy)\Big)^{p/2}\Big]$$
$$\leqslant \sum_{k=0,\dots,2^n-2} \mathbb{E}\Big[\Big(\int_{x,y\in[\frac{k}{2^n},\frac{k+2}{2^n}]^2} \frac{1}{|x|^\delta} M(dx)M(dy)\Big)^{p/2}\Big]$$
$$= \mathbb{E}\Big[\Big(\int_{x,y\in[0,2^{-n+1}]^2} \frac{1}{|x|^\delta} M(dx)M(dy)\Big)^{p/2}\Big] + \sum_{k=1,\dots,2^n-2} \frac{2^{n\delta}}{k^\delta} \mathbb{E}\Big[M\Big([\frac{k}{2^n}, \frac{k+2}{2^n}]\Big)^p\Big]$$

By stationarity and scale invariance, we get:

$$\sum_{k=1,\dots,2^n-2} \frac{2^{n\delta}}{k^\delta} \mathbb{E}\Big[M\Big([\frac{k}{2^n}, \frac{k+2}{2^n}]\Big)^p\Big]$$
$$\leqslant 2^{n\delta} \sum_{k=1,\dots,2^n-2} \frac{1}{k^\delta} \mathbb{E}\Big[M\Big([0, 2^{-n+1}]\Big)^p\Big]$$
$$\leqslant 2^{n\delta} 2^{-(n-1)\xi_M(p)} \mathbb{E}\Big[M\Big([0,1]\Big)^p\Big] \sum_{k=1,\dots,2^n-2} \frac{1}{k^\delta}$$
$$\leqslant \frac{C}{1-\delta} 2^{-n(\xi_M(p)-1)}$$

where C only depends on $\mathbb{E}\big[M\big([0,1]\big)^p\big]$. We conclude with Lemma B.2. $\square$

Proof of Lemma B.1. Choose another δ' such that $0 < \delta' + \delta < \frac{2(\xi_M(p)-2)}{p}$. By using Lemma B.3, we have:

$$\mathbb{E}\Big[\Big(\int_{\max(|x|,|y|) \leqslant 1} \frac{1}{|x|^\delta |x-y|^{\delta'}} M(dx)M(dy)\Big)^{p/2}\Big]$$
$$= \mathbb{E}\Big[\Big(\sum_{n=0}^{+\infty} \int_{\substack{\max(|x|,|y|) \leqslant 1 \\ 2^{-n-1} \leqslant |x-y| \leqslant 2^{-n}}} \frac{1}{|x|^\delta |x-y|^{\delta'}} M(dx)M(dy)\Big)^{p/2}\Big]$$
$$\leqslant \sum_{n=0}^{+\infty} \mathbb{E}\Big[\Big(\int_{\substack{\max(|x|,|y|) \leqslant 1 \\ 2^{-n-1} \leqslant |x-y| \leqslant 2^{-n}}} \frac{1}{|x|^\delta |x-y|^{\delta'}} M(dx)M(dy)\Big)^{p/2}\Big]$$
$$\leqslant \sum_{n=0}^{+\infty} 2^{(n+1)\frac{\delta' p}{2}} \mathbb{E}\Big[\Big(\int_{\substack{\max(|x|,|y|) \leqslant 1 \\ 2^{-n-1} \leqslant |x-y| \leqslant 2^{-n}}} \frac{1}{|x|^\delta} M(dx)M(dy)\Big)^{p/2}\Big]$$
$$\leqslant \sum_{n=0}^{+\infty} 2^{(n+1)\frac{\delta' p}{2}} C 2^{-n(\xi_M(p)-2-\delta p/2)}.$$

Since the latter series converges, the proof is complete. $\square$

Now we prove that the function f satisfies:

$$\mathbb{E}^X\Big[\Big(\int_{|x|\leqslant 1,|x-y|\leqslant 1} f(x,y)M(dx)M(dy)\Big)^{p/2}\Big] < +\infty.$$

To this purpose, it is enough to prove that the divergence of f when approaching the diagonal is logarithmic:

Lemma B.4. *We claim:*

$$f(x,y) \leqslant D(1+\ln_+ \frac{1}{|x-y|})(1+\ln_+ \frac{1}{|x|}),$$

for some constant D that only depends on t.

Proof. Recall that:

$$f(x,y) = \int_0^t \int_s^t e^{-\frac{|x|^2}{2s} - \frac{|y-x|^2}{2|r-s|}} \frac{drds}{4\pi^2 s|r-s|}.$$

By making successive changes of variables, we obtain:

$$\begin{aligned} f(x,y) &= \int_0^t \int_0^{\frac{t-s}{|x-y|^2}} e^{-\frac{|x|^2}{2s} - \frac{1}{2r}} \frac{drds}{4\pi^2 sr} \\ &= \int_0^{\frac{t}{|x|^2}} \int_0^{\frac{t-s|x|^2}{|x-y|^2}} e^{-\frac{1}{2s} - \frac{1}{2r}} \frac{drds}{4\pi^2 sr} \\ &\leqslant g(\frac{t}{|x|^2}) g(\frac{t}{|x-y|^2}), \end{aligned}$$

where we have set

$$g(t) = \int_0^t e^{-\frac{1}{2s}} \frac{ds}{2\pi s}.$$

It is obvious to check that, for some constant $D>0$, we have

$$g(t) \leqslant D(1+\ln_+ t).$$

The proof is complete. $\square$

By gathering Lemma B.4 and Lemma B.1, we deduce

Corollary B.5.

$$\mathbb{E}^X\Big[\Big(\int_{|x|\leqslant 1,|x-y|\leqslant 1} f(x,y)M(dx)M(dy)\Big)^{p/2}\Big] < +\infty.$$

Domain $\{|x-y| \geqslant 1\}$

Let us now investigate the situation when $|x-y| \geqslant 1$. This is the easy part because, in that case, the measures $M(dx)$ and $M(dy)$ are "almost" independent. Therefore, we can proceed more directly in the computations. We claim:

Lemma B.6. *The following expectation is finite:*

$$\mathbb{E}^X\Big[\Big(\int_{|x-y| \geqslant 1} f(x,y)M(dx)M(dy)\Big)^{p/2}\Big] < +\infty.$$

Proof. We use the Jensen inequality with the concave function $x \mapsto x^{p/2}$ to get:

$$\begin{aligned}
&\mathbb{E}^X\Big[\Big(\int_{|x-y| \geqslant 1} f(x,y)M(dx)M(dy)\Big)^{p/2}\Big] \\
&\qquad \leqslant \Big(\mathbb{E}^X\Big[\int_{|x-y| \geqslant 1} f(x,y)M(dx)M(dy)\Big]\Big)^{p/2} \\
&\qquad \leqslant \Big(\int_{|x-y| \geqslant 1} f(x,y)e^{\gamma^2 G_m(x,y)}\,dxdy\Big)^{p/2}.
\end{aligned}$$

Since $|x-y| \geqslant 1$, we have $G_m(x,y) \leqslant C$ for some fixed positive constant C. We deduce:

$$\begin{aligned}
&\mathbb{E}^X\Big[\Big(\int_{|x-y| \geqslant 1} f(x,y)M(dx)M(dy)\Big)^{p/2}\Big] \\
&\qquad \leqslant e^{Cp/2}\Big(\int_{\mathbb{R}^2\times\mathbb{R}^2} f(x,y)\,dxdy\Big)^{p/2} \\
&\qquad = e^{Cp/2}.
\end{aligned}$$

We complete the proof of the lemma. $\square$

Domain $\{|x| \geqslant 1, |x-y| \leqslant 1\}$

The final part of the proof consists in checking the following lemma:

Lemma B.7. *The following expectation is finite:*

$$\mathbb{E}^X\Big[\Big(\int_{|x| \geqslant 1, |x-y| \leqslant 1} f(x,y)M(dx)M(dy)\Big)^{p/2}\Big] < +\infty.$$

To prove this lemma, the first step consists in identifying the behaviour of f on the domain $\{|x| \geqslant 1, |x-y| \leqslant 1\}$. Following the lines of Lemma B.4, the reader may check the following lemma:

Lemma B.8. *There exists a constant $D > 0$ such that*

$$f(x,y) \leqslant D(1 + \ln_+ \frac{1}{|x-y|}) \exp\big(-\frac{|x|^2}{2t}\big),$$

for all $|x| \geqslant 1$ and $|x-y| \leqslant 1$.

Therefore, the proof of Lemma B.7 just boils down to proving:

Lemma B.9. *Fix $t > 0$. For any $\gamma^2 < 4$ and $p \in]1, \frac{4}{\gamma^2}[$, there exist $\delta > 0$ and a constant $C > 0$ (only depending on $\mathbb{E}\big[M\big([0,1]\big)^p\big]$) such that:*

$$\mathbb{E}\Big[\Big(\int_{\substack{|x| \geqslant 1\\ |x-y| \leqslant 1}} \frac{\exp\big(-\frac{|x|^2}{2t}\big)}{|x-y|^\delta} M(dx)M(dy)\Big)^{p/2}\Big] < +\infty.$$

Proof. We go on carrying out the proof in dimension 1 to avoid notational issues. Once again, we first need to estimate the above expectation on stripes of the type $\{|x| \geqslant 1, 2^{-n-1} \leqslant |x-y| \leqslant 2^{-n}\}$. So we claim:

Lemma B.10. *Fix $t > 0$. For any $\gamma^2 < 4$ and $p \in]1, \frac{4}{\gamma^2}[$, there exists a constant $C > 0$ (only depending on $\mathbb{E}\big[M\big([0,1]\big)^p\big]$) such that for all n:*

$$\mathbb{E}\Big[\Big(\int_{\substack{|x| \geqslant 1\\ 2^{-n-1} \leqslant |x-y| \leqslant 2^{-n}}} \exp\big(-\frac{|x|^2}{2t}\big) M(dx)M(dy)\Big)^{p/2}\Big] \leqslant C2^{-n(\xi_M(p)-2)}.$$

Let us admit for a while the above lemma to finish the proof of Lemma B.9. Choose δ such that $0 < \delta < \frac{2(\xi_M(p)-2)}{p}$. By using Lemma B.10, we have:

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{\substack{|x| \geqslant 1\\ |x-y| \leqslant 1}} \frac{\exp\big(-\frac{|x|^2}{2t}\big)}{|x-y|^\delta} M(dx)M(dy)\Big)^{p/2}\Big]\\
&\quad = \mathbb{E}\Big[\Big(\sum_{n=0}^{+\infty} \int_{\substack{|x| \geqslant 1\\ 2^{-n-1} \leqslant |x-y| \leqslant 2^{-n}}} \frac{\exp\big(-\frac{|x|^2}{2t}\big)}{|x-y|^\delta} M(dx)M(dy)\Big)^{p/2}\Big]\\
&\quad \leqslant \sum_{n=0}^{+\infty} \mathbb{E}\Big[\Big(\int_{\substack{|x| \geqslant 1\\ 2^{-n-1} \leqslant |x-y| \leqslant 2^{-n}}} \frac{\exp\big(-\frac{|x|^2}{2t}\big)}{|x-y|^\delta} M(dx)M(dy)\Big)^{p/2}\Big]\\
&\quad \leqslant \sum_{n=0}^{+\infty} 2^{(n+1)\frac{\delta p}{2}} \mathbb{E}\Big[\Big(\int_{\substack{|x| \geqslant 1\\ 2^{-n-1} \leqslant |x-y| \leqslant 2^{-n}}} \exp\big(-\frac{|x|^2}{2t}\big) M(dx)M(dy)\Big)^{p/2}\Big]\\
&\quad \leqslant \sum_{n=0}^{+\infty} 2^{(n+1)\frac{\delta p}{2}} C 2^{-n(\xi_M(p)-2)}.
\end{aligned}$$

Since the latter series converges, the proof is complete. $\square$

Proof of Lemma B.10. Once again the choice of the log-correlated Gaussian field is left open thanks to Kahane's convexity inequalities and we choose the perfect kernel of [7]. It is also plain to check that the expectation is finite thanks to the exponential term. We will prove the result when integrating only over the domain $\{x \geqslant 1, 2^{-n-1} \leqslant |x-y| \leqslant 2^{-n}\}$. It will then be obvious to complete the proof (for instance by using invariance of M in law under reflection). As previously, the reader may check that the stripe $\{x \geqslant 1, 2^{-n-1} \leqslant |x-y| \leqslant 2^{-n}\}$ may be covered by the squares $[\frac{k}{2^n}, \frac{k+2}{2^n}]^2$ for k running over the set $K_n = \mathbb{Z} \cap [2^n, +\infty[$. Therefore, by using sub-additivity of the mapping $x \mapsto x^{p/2}$, we have:

$$\begin{aligned}
\mathbb{E}\Big[\Big(\int_{\substack{x \geqslant 1 \\ 2^{-n-1} \leqslant |x-y| \leqslant 2^{-n}}} & \exp\big(-\frac{|x|^2}{2t}\big) M(dx)M(dy)\Big)^{p/2}\Big] \\
& \leqslant \mathbb{E}\Big[\Big(\sum_{k \in K_n} \int_{[\frac{k}{2^n}, \frac{k+2}{2^n}]^2} \exp\big(-\frac{|x|^2}{2t}\big) M(dx)M(dy)\Big)^{p/2}\Big] \\
& \leqslant \sum_{k \in K_n} \mathbb{E}\Big[\Big(\int_{[\frac{k}{2^n}, \frac{k+2}{2^n}]^2} \exp\big(-\frac{k^2}{t2^{2n+1}}\big) M(dx)M(dy)\Big)^{p/2}\Big] \\
& = \sum_{k \in K_n} \exp\big(-\frac{k^2 p}{t2^{2n+2}}\big) \mathbb{E}\Big[M\Big([\frac{k}{2^n}, \frac{k+2}{2^n}]\Big)^p\Big]
\end{aligned}$$

By stationarity and scale invariance, we get:

$$\begin{aligned}
\sum_{k \in K_n} \exp\big(-\frac{k^2 p}{t2^{2n+2}}\big) & \mathbb{E}\Big[M\Big([\frac{k}{2^n}, \frac{k+2}{2^n}]\Big)^p\Big] \\
& = \sum_{k \in K_n} \exp\big(-\frac{k^2 p}{t2^{2n+2}}\big) \mathbb{E}\Big[M\Big([0, 2^{n-1}]\Big)^p\Big] \\
& = 2^{-(n-1)\xi_M(p)} \sum_{k \in K_n} \exp\big(-\frac{k^2 p}{t2^{2n+2}}\big) \mathbb{E}\Big[M\Big([0,1]\Big)^p\Big] \\
& \leqslant C 2^{-n(\xi_M(p)-1)}.
\end{aligned}$$

where C only depends on $\mathbb{E}\big[M\big([0,1]\big)^p\big]$. The last line uses the standard trick of convergence of Riemann sums. $\square$

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